Solutions

Midterm Exam No. 02 (Fall 2014)

PHYS 320: Electricity and Magnetism I

Date: 2014 Oct 15

1. **(20 points.)** Electric field lines due to four positive charges of equal magnitude placed at the vertices of a square are drawn in Fig. 1. Using Fig. 1 as a guide, estimate the approximate coordinates (choosing the red dot as origin) of *all* the points where a test charge will not experience a force. Also, comment on the stability (or instability) of a test charge kept at these points.

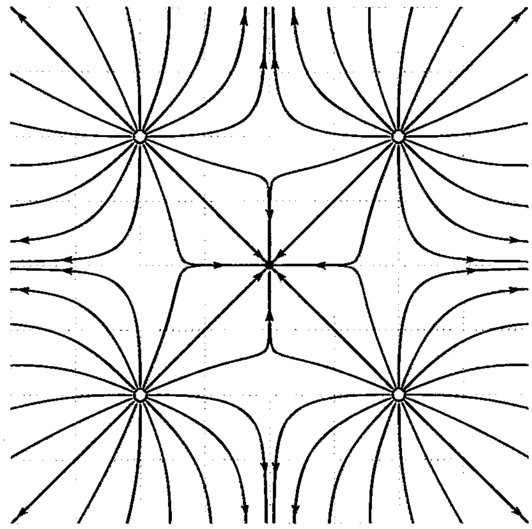


Figure 1: Problem 1.

2. **(20 points.)** The electric field of a point dipole $\mathbf{p}$ is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\Big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\Big] = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2}{r^5}\right]. \tag{1}$$

Evaluate $\boldsymbol{\nabla}\cdot\mathbf{E}$.
Hint: Intuitively, the divergence of a vector field is a measure of the density of source/sink of the field. For reference, the electric field lines drawn in Fig. 2 will be those of a point dipole in the limit of distance between the two charges going to zero, keeping the magnitude of the dipole moment fixed.

3. **(20 points.)** Earnshaw's theorem states that Poisson equation does not allow stable configurations for electric field going to zero at infinity. After the lecture on Earnshaw's

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

6. **(20 points.)** Consider an infinitely thin flat sheet, of infinite extent, constructed out of a continuous distribution of point dipoles, each of individual charge density

$$-\mathbf{p} \cdot \boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}_p), \tag{4}$$

where $\mathbf{r}_p$ is the position of an individual point dipole. The charge density of such a sheet is given by

$$\rho(\mathbf{r}) = -\sigma \frac{\partial}{\partial z}\delta(z), \tag{5}$$

where σ is the electric dipole moment per unit area.

(a) Evaluate the electric potential for the sheet using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{6}$$

(Hint: Use the δ-function property to evaluate the z'-integral, after integrating by parts. The x' and y' integrals can be completed using standard substitutions.)

(b) Evaluate the electric field for the sheet by finding the gradient of the electric potential you calculated using Eq. (6),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{7}$$

MT02, prob 1

(i) $(0, 0, 0)$

(ii) $(+0.7734, 0, 0)$

(iii) $(-0.7734, 0, 0)$

(iv) $(0, +0.7734, 0)$

(v) $(0, -0.7734, 0)$

→ All are unstable points.

→ Assumed charges to be at $(1,1,0)$, $(1,-1,0)$, $(-1,1,0)$, $(-1,-1,0)$.

MT02, prob 2

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p}\cdot\vec{r})\,\vec{r} - \vec{p}\,r^2}{r^5}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0} \vec{\nabla}\cdot\left(3\,\frac{(\vec{p}\cdot\vec{r})\,\vec{r}}{r^5}\right) - \frac{1}{4\pi\epsilon_0}\vec{\nabla}\cdot\left(\frac{\vec{p}}{r^3}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\, 3\, \frac{1}{r^5}\, \vec{r}\cdot\vec{\nabla}(\vec{p}\cdot\vec{r}) + \frac{1}{4\pi\epsilon_0}\, 3\, \frac{(\vec{p}\cdot\vec{r})}{r^5} \underbrace{(\vec{\nabla}\cdot\vec{r})}_{=3} + \frac{1}{4\pi\epsilon_0}\, 3(\vec{p}\cdot\vec{r})\, \vec{r}\cdot\vec{\nabla}\left(\frac{1}{r^5}\right)$$

$$- \frac{1}{4\pi\epsilon_0} \frac{(\cancelto{0}{\vec{\nabla}\cdot\vec{p}})}{r^3} - \frac{1}{4\pi\epsilon_0}\, \vec{p}\cdot\vec{\nabla}\frac{1}{r^3}$$

$$= \frac{3}{4\pi\epsilon_0} \frac{1}{r^5}\, \vec{r}\cdot \underbrace{\vec{\nabla}(\vec{r}}_{=\overleftrightarrow{1}}\cdot\vec{p}) + \frac{9}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5} + \frac{3}{4\pi\epsilon_0}(\vec{p}\cdot\vec{r})\,\vec{r}\cdot\frac{(-5)}{r^6}\underbrace{(\vec{\nabla}r)}_{=\hat{r}}$$

$$- \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{(-3)}{r^4}\underbrace{(\vec{\nabla}r)}_{=\hat{r}}$$

$$= \frac{3}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5} + \frac{9}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5} - \frac{15}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5} + \frac{3}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5}$$

$$= 0$$

MT02, prob 3

Electric field is zero inside a uniformly charged spherical shell. Thus, a small displacement about the origin does not change the force. So, the origin is neither a stable point nor an unstable point.

MT02, prob 4

(a)

$$Q = \int d^3r\, \rho(\vec{r}) = \int d^3r\, b\, r^3\, \theta(R-r)$$

$$= \int_0^\infty r^2 dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{4\pi}\, b\, r^3\, \theta(R-r)$$

$$= 4\pi b \int_0^R dr\, r^5$$

$$= 4\pi b \frac{R^6}{6} = \frac{2\pi}{3} b R^6$$

$$\Rightarrow \quad b = \frac{3}{2\pi}\frac{Q}{R^6}$$

(b)

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{in}$$

$$E\, 4\pi r^2 = \frac{1}{\epsilon_0} Q_{in}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2} Q_{in}$$

$\underline{\underline{r<R}}$:

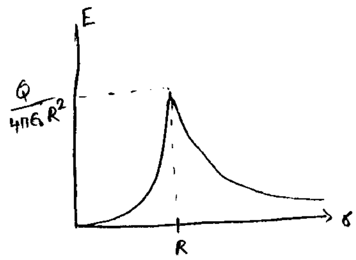
$$Q_{in} = \int_0^r r'^2 dr'\, 4\pi\, b\, r'^3\, \theta(R-r')$$

$$= 4\pi b \int_0^r dr'\, r'^5 = 4\pi b \frac{r^6}{6} = \frac{2\cancel{\pi}}{\cancel{3}}\frac{\cancel{3}}{2\cancel{\pi}}\frac{Q}{R^6} r^6 = \frac{r^6}{R^6} Q$$

$\underline{\underline{r>R}}$: $\quad Q_{in} = Q.$

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{R^2}\left(\dfrac{r}{R}\right)^4, & \text{if } r<R, \\ \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r^2}, & \text{if } R<r. \end{cases}$$

(c)



MT02, prob 5

$$y = r^2 + r'^2 - 2rr't$$
$$dy = -2rr'dt$$

$$I = \frac{1}{2}\int_{-1}^{1} dt \, \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}$$

$$= \frac{1}{2}\int_{(r+r')^2}^{(r-r')^2} (-1)\,\frac{dy}{2rr'}\,\frac{1}{\sqrt{y}}$$

$$= \frac{1}{4rr'}\int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{4rr'}\; 2\sqrt{y}\,\Big|_{y=(r-r')^2}^{y=(r+r')^2}$$

$$= \frac{1}{2rr'}\Big[(r+r') - |r-r'|\Big]$$

$$= \begin{cases} \frac{1}{2rr'}\left[\cancel{r} + r' - \cancel{r} + r'\right] & \text{if } r > r' \\ \frac{1}{2rr'}\left[r + \cancel{r'} - \cancel{r'} + r\right] & \text{if } r' > r \end{cases}$$

$$= \begin{cases} \frac{1}{r}, & \text{if } r > r' \\ \frac{1}{r'}, & \text{if } r' > r \end{cases} \qquad r_> \equiv \text{Max}(r, r')$$

$$= \frac{1}{r_>}$$

MT02, prob 6

$$\rho(\vec{r}) = -\sigma \frac{\partial}{\partial z} \delta(z)$$

(a)
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} (-\sigma) \frac{\partial}{\partial z'} \delta(z')$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \left[\frac{\partial}{\partial z'} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \right] \delta(z') + \text{surface term} \hookrightarrow = 0$$

$$= -\frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \frac{z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{\frac{3}{2}}}$$

$x - x' = \rho \cos\phi$
$y - y' = \rho \sin\phi$
$\Rightarrow dx'dy' = \rho\, d\rho\, d\phi$

$$= -\frac{\sigma}{4\pi\epsilon_0} z \int_0^{\infty} \rho\, d\rho \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \frac{1}{(\rho^2+z^2)^{\frac{3}{2}}}$$

$\rho^2 + z^2 = t$
$2\rho\, d\rho = dt$

$$= -\frac{\sigma}{2\epsilon_0} z \int_0^{\infty} \rho\, d\rho \frac{1}{(\rho^2+z^2)^{\frac{3}{2}}}$$

$$\frac{t^{-\frac{3}{2}+1}}{2\left(-\frac{1}{2}\right)} = -\frac{1}{\sqrt{t}}$$

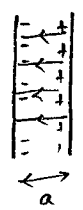
$$= -\frac{\sigma}{2\epsilon_0} z \int_{\rho^2}^{\infty} \frac{dt}{2} \frac{1}{t^{3/2}}$$

$$= \frac{\sigma}{2\epsilon_0} z \left[\frac{1}{\infty} - \frac{1}{z} \right]$$

$$= -\frac{\sigma}{2\epsilon_0}$$

(b) $\vec{E} = -\vec{\nabla}\phi = 0$

Intuitively, if we had dipoles with non-zero distances we have.

E = 0  E = 0 $\xrightarrow[a \to 0]{}$ $\vec{E} = 0$ everywhere.