→ Read hints.

Midterm Exam No. 03 (Fall 2014)

PHYS 320: Electricity and Magnetism I

Date: 2014 Nov 7

Solutions

1. **(20 points.)** The force and torque on an electric dipole $\mathbf{d}$ in the presence of an electric field is given by

(very short, < 2 steps.)

$$\mathbf{F} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E} \qquad \text{and} \qquad \boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}, \tag{1}$$

respectively. Thus, describe the motion of an electric dipole when placed in between the plates of a parallel plate capacitor. Assume the plates to be perfectly conducting, of infinite cross-sectional area, and the medium in between to be vacuum.

2. **(20 points.)** Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by

(very short, < 3 steps.)

$$\mathbf{P}(\mathbf{r}) = \sigma\, \hat{\mathbf{x}}\, \theta(-z), \tag{2}$$

where σ is the polarization per unit area of the slab. Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P}. \tag{3}$$

What can you say about the electric field, inside and outside the slab?

3. **(20 points.)** (Based on problem 4.13 Griffiths 4th/3rd edition.)
Consider a solid cylinder of radius R, and infinite length, with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\, \theta(R - \rho), \tag{4}$$

where $\rho^2 = x^2 + y^2$. Here the uniform polarization $\mathbf{P}_0$, constant in space and time, is presumed to be perpendicular to the axis of the cylinder.

(a) Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P}. \tag{5}$$

(b) The electric potential outside the cylinder, $R < \rho$, is given by, (need not be derived here,)

$$\phi(\mathbf{r}) = \frac{R^2}{2\varepsilon_0} \frac{\mathbf{P_0} \cdot \hat{\boldsymbol{\rho}}}{\rho}, \tag{6}$$

where $\hat{\boldsymbol{\rho}}$ is the unit vector along the cylindrical polar coordinate ρ. Determine the electric field outside the cylinder, $R < \rho$, by evaluating the gradient of the electric potential,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{7}$$

(a) The electric field of a point charge Q at distance $\mathbf{R}$ from the charge is

$$\mathbf{E}(\mathbf{R}) = \frac{Q}{4\pi\varepsilon_0} \frac{\mathbf{R}}{R^3}. \tag{12}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d} \cdot \mathbf{E}. \tag{13}$$

Thus, derive the interaction energy between the charge Q and the dipole $\mathbf{d}$ to be

$$U = -\frac{Q}{4\pi\varepsilon_0} \frac{\mathbf{d} \cdot \mathbf{R}}{R^3}. \tag{14}$$

(b) The variables in the problem are the coordinate ϕ that specifies the position of the dipole on the circular track, and the angle θ that the direction of the dipole makes with respect to the radius vector $\mathbf{R}$. Thus, conclude that the interaction energy is independent of the coordinate ϕ,

$$U(\theta) = -\frac{Q}{4\pi\varepsilon_0} \frac{d \cos\theta}{R^2}. \tag{15}$$

(c) The generalized tangential force on the dipole, upto a factor R, is

$$F_\phi = -\frac{\partial}{\partial \phi} U. \tag{16}$$

Thus, conclude that there is no tangential force acting on the dipole. No perpetual motion!

(d) The torque acting on the dipole is

$$F_\theta = -\frac{\partial}{\partial \theta} U. \tag{17}$$

Determine the angles for which this force in zero. Analyse each of these angles and find out if they are stable or unstable.

(e) Describe the motion of the dipole on the track for arbitrary initial conditions with respect to ϕ and θ. That is, describe your results in 5d.

MT-03, prob. 1

Electric field is constant inside a parallel plate capacitor.

Thus,
$$\vec{F} = 0$$
$$\vec{\tau} = \vec{d} \times \vec{E}$$

The torque will try to align $\vec{d}$ with $\vec{E}$, which will cause an oscillatory motion.

MT-03, prob 2

$$\rho_{eff} = -\vec{\nabla}\cdot\vec{P}$$
$$= -\vec{\nabla}\cdot\left[\tau\,\hat{x}\,\theta(-z)\right]$$
$$= -\tau\,\hat{x}\cdot\vec{\nabla}\,\theta(-z)$$
$$= -\tau\,\frac{\partial}{\partial x}\,\theta(-z)$$
$$= 0.$$

$$\hat{x}\cdot\vec{\nabla} = \frac{\partial}{\partial x}.$$

Since
$$\vec{\nabla}\cdot\epsilon_0\vec{E} = -\vec{\nabla}\cdot\vec{P} = 0$$

Thus, $\vec{E}$ is a constant, with no abrupt change at the boundary. It is as well zero everywhere.

MT-03, prob 3

(a)

$$\rho_{eff}(\vec{r}) = -\vec{\nabla}\cdot\vec{P}$$

$$= -\vec{\nabla}\cdot\left[\vec{P}_0\ \theta(R-\rho)\right]$$

$$= -\vec{P}_0\cdot\vec{\nabla}\ \theta(R-\rho)$$

$$\vec{\nabla}\rho = \hat{\rho}$$

$$= -\vec{P}_0\cdot(\vec{\nabla}\rho)\ \frac{\partial}{\partial\rho}\theta(R-\rho)$$

$$= \vec{P}_0\cdot(\vec{\nabla}\rho)\ \delta(R-\rho)$$

$$= \vec{P}_0\cdot\hat{\rho}\ \delta(R-\rho)$$

(b)

$$\phi(\vec{r}) = \frac{R^2}{2\epsilon_0}\ \frac{\vec{P}_0\cdot\hat{\rho}}{\rho} = \frac{R^2}{2\epsilon_0}\ \frac{\vec{P}_0\cdot\vec{\rho}}{\rho^2}$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\frac{R^2}{2\epsilon_0}\ \vec{\nabla}\left[\frac{\vec{P}_0\cdot\vec{\rho}}{\rho^2}\right]$$

$$= -\frac{R^2}{2\epsilon_0}\left[\frac{(\vec{\nabla}\vec{\rho})\cdot\vec{P}_0}{\rho^2} + (\vec{P}_0\cdot\vec{\rho})\ \frac{(-2)}{\rho^3}\ \vec{\nabla}\rho\right]$$

$$\vec{\nabla}\vec{\rho} = \hat{x}\hat{x} + \hat{y}\hat{y} \Rightarrow (\vec{\nabla}\vec{\rho})\cdot\vec{P}_0 = \vec{P}_0$$

$$\vec{\nabla}\rho = \hat{\rho}$$

$$\vec{E} = -\frac{R^2}{2\epsilon_0}\left[\frac{\vec{P}_0}{\rho^2} - 2\ \frac{(\vec{P}_0\cdot\vec{\rho})\ \hat{\rho}}{\rho^3}\right]$$

$$= \frac{R^2}{2\epsilon_0}\ \frac{1}{\rho^2}\left[2\ (\vec{P}_0\cdot\hat{\rho})\ \hat{\rho} - \vec{P}_0\right]$$

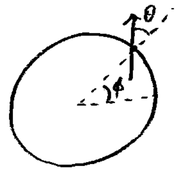
MT-03, prob 4

$$
\begin{aligned}
\rho_{eff} &= -\vec{\nabla}\cdot\vec{P} \\
&= -\vec{\nabla}\cdot\left[\vec{P}_0\ \theta(\mu_R-\mu)\right] \\
&= -\vec{P}_0\cdot\vec{\nabla}\ \theta(\mu_R-\mu) \\
&= -\vec{P}_0\cdot(\vec{\nabla}\mu)\frac{\partial}{\partial\mu}\theta(\mu_R-\mu) \\
&= \vec{P}_0\cdot(\vec{\nabla}\mu)\ \delta(\mu_R-\mu) \\
&= \vec{P}_0\cdot\hat{\mu}\ \delta(\mu_R-\mu)
\end{aligned}
$$

$\hat{\mu} = \vec{\nabla}\mu$

MT-03, prob. 5

(a) $U = -\vec{d}\cdot\vec{E} = -\dfrac{Q}{4\pi\epsilon_0}\dfrac{\vec{d}\cdot\vec{R}}{R^3}$

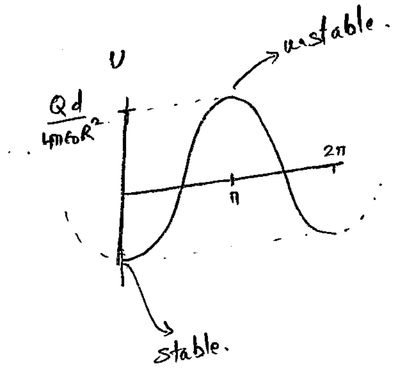


(b) Using $\vec{d}\cdot\vec{R} = dR\cos\theta$.

$$U(\theta) = -\frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}$$

(c) $F_\phi = -\dfrac{\partial}{\partial\phi}U$

$= 0$.



(d) $F_\theta = -\dfrac{\partial}{\partial\theta}U$

$= -\dfrac{Q}{4\pi\epsilon_0}\dfrac{d\sin\theta}{R^2}$

$F_\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$

$$\frac{\partial^2 U}{\partial\theta^2} = \frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}$$

$$\left.\frac{\partial^2 U}{\partial\theta^2}\right|_{\theta=0} = \frac{Qd}{4\pi\epsilon_0 R^2} > 0 \Rightarrow \text{stable point}$$

$$\left.\frac{\partial^2 U}{\partial\theta^2}\right|_{\theta=\pi} = -\frac{Qd}{4\pi\epsilon_0 R^2} < 0 \Rightarrow \text{unstable point.}$$

(e) $\phi \rightarrow$ It will eventually come to rest w.r.t. ϕ, because there is no centripetal force to fuel it.

$\theta \rightarrow$ It will oscillate with respect to θ, and $\vec{d}$ will try to align with $\hat{R}$.