Midterm Exam No. 03 (Fall 2014)

PHYS 320: Electricity and Magnetism I

Date: 2014 Nov 7

1. (**20 points.**) The force and torque on an electric dipole $\mathbf{d}$ in the presence of an electric field is given by
$$\mathbf{F} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E} \qquad \text{and} \qquad \boldsymbol{\tau} = \mathbf{d} \times \mathbf{E}, \tag{1}$$
respectively. Thus, describe the motion of an electric dipole when placed in between the plates of a parallel plate capacitor. Assume the plates to be perfectly conducting, of infinite cross-sectional area, and the medium in between to be vacuum.

2. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by
$$\mathbf{P}(\mathbf{r}) = \sigma\, \hat{\mathbf{x}}\, \theta(-z), \tag{2}$$
where σ is the polarization per unit area of the slab. Determine the effective charge density by evaluating
$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P}. \tag{3}$$
What can you say about the electric field, inside and outside the slab?

3. (**20 points.**) (Based on problem 4.13 Griffiths 4th/3rd edition.)
Consider a solid cylinder of radius R, and infinite length, with uniform permanent polarization
$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\, \theta(R - \rho), \tag{4}$$
where $\rho^2 = x^2 + y^2$. Here the uniform polarization $\mathbf{P}_0$, constant in space and time, is presumed to be perpendicular to the axis of the cylinder.

   (a) Determine the effective charge density by evaluating
   $$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P}. \tag{5}$$

   (b) The electric potential outside the cylinder, $R < \rho$, is given by, (need not be derived here,)
   $$\phi(\mathbf{r}) = \frac{R^2}{2\varepsilon_0} \frac{\mathbf{P_0} \cdot \hat{\boldsymbol{\rho}}}{\rho}, \tag{6}$$
   where $\hat{\boldsymbol{\rho}}$ is the unit vector along the cylindrical polar coordinate ρ. Determine the electric field outside the cylinder, $R < \rho$, by evaluating the gradient of the electric potential,
   $$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{7}$$

Hint: You will need to evaluate, or be aware of the results for, the dyadic $\boldsymbol{\nabla}\boldsymbol{\rho}$ and the vector $\boldsymbol{\nabla}\rho$. Most of the concepts and results for the uniformly polarized sphere, that was covered in class, repeats reasonably closely for a cylinder.

4. (**20 points.**) Consider a uniformly polarized cylinder, of elliptic cross-section, described by

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\,\theta(\mu_R - \mu), \tag{8}$$

in terms of the elliptic coordinates (μ, ν) defined as

$$x = a\cosh\mu\cos\nu, \tag{9}$$

$$y = a\sinh\mu\sin\nu, \tag{10}$$

where $\mu \geq 0$ parameterizes confocal ellipses, $0 \leq \nu < 2\pi$ parameterizes confocal hyperbolae, such that $x = \pm a$ are the two foci of the ellipse. Thus, μ_R specifies the particular confocal ellipse. Evaluate the effective charge density

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{11}$$

for the polarized ellipse in terms of the elliptic coordinates and the respective unit vectors. Hint: Unit vectors are given by the gradient of the respective coordinate surfaces. The answer to this question does not require a detailed calculation. It conceptually follows the analogous problem for spherical geometry at every step.

5. (**20 points. Take home.**) Here is problem 4.31 of Griffiths 4th edition, which is not there in the 3rd edition:

> A point charge Q is "nailed down" on a table. Around it, at radius R is a frictionless circular track on which a dipole $\mathbf{d}$ rides, constrained always to point tangent to the circle. Use Eq. (4.5) of Griffiths, 4th/3rd edition, to show that the electric force on the dipole is
>
> $$\mathbf{F} = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}}{R^3}.$$
>
> Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

Footnote 21, in Griffiths 4th edition, is an acknowledgement: "This charming paradox was suggested by K. Brownstein."
You might also refer to comments by Prof. Alan Guth, in his Fall 2014 lecture notes, at
`http://web.mit.edu/8.07/www/probsets/ps06-f14.pdf`
`http://web.mit.edu/8.07/www/probsets/sol06-f14.pdf`

(a) The electric field of a point charge Q at distance $\mathbf{R}$ from the charge is

$$\mathbf{E}(\mathbf{R}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{R}}{R^3}. \tag{12}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d}\cdot\mathbf{E}. \tag{13}$$

Thus, derive the interaction energy between the charge Q and the dipole $\mathbf{d}$ to be

$$U = -\frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{R}}{R^3}. \tag{14}$$

(b) The variables in the problem are the coordinate ϕ that specifies the position of the dipole on the circular track, and the angle θ that the direction of the dipole makes with respect to the radius vector $\mathbf{R}$. Thus, conclude that the interaction energy is independent of the coordinate ϕ,

$$U(\theta) = -\frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{R^2}. \tag{15}$$

(c) The generalized tangential force on the dipole, upto a factor R, is

$$F_\phi = -\frac{\partial}{\partial\phi}U. \tag{16}$$

Thus, conclude that there is no tangential force acting on the dipole. No perpetual motion!

(d) The torque acting on the dipole is

$$F_\theta = -\frac{\partial}{\partial\theta}U. \tag{17}$$

Determine the angles for which this force in zero. Analyse each of these angles and find out if they are stable or unstable.

(e) Describe the motion of the dipole on the track for arbitrary initial conditions with respect to ϕ and θ. That is, describe your results in 5d.