Solutions

Final Exam (Fall 2014)

PHYS 320: Electricity and Magnetism I

Date: 2014 Dec 10

1. **(20 points.)** The electric field due to a point dipole $\mathbf{d}$ at a distance $\mathbf{r}$ away from dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}\big]. \tag{1}$$

   Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{d} = d\,\hat{\mathbf{z}}$.

   (a) Qualitatively plot the electric field lines for the dipole $\mathbf{d}$. (It should be a good sketch.)

   (b) Find the (simplified) expression for the electric field on the positive z-axis. (Hint: On the positive z-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = z$.)

2. **(20 points.)** Evaluate the integral

$$\int_{-1}^{1} dx\,\delta(1-3x)\Big[9x^2 + 3x - 1\Big]. \tag{2}$$

3. **(20 points.)** Consider a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density

$$\rho(\mathbf{r}) = br\,\theta(R-r), \tag{3}$$

   where r is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.

   (a) Integrating the charge density over all space gives you the total charge Q. Thus, determine the constant b in terms of Q and R.

   (b) Using Gauss's law find the electric field inside and outside the sphere.

   (c) Plot the magnitude of the electric field with respect to r.

4. **(20 points.)** Consider a right circular cone with uniform polarization $\mathbf{P}_0$, of infinite height, apex at the origin, aperture angle $2\theta_0$, described by

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\,\theta_{\text{fun}}(\theta_0 - \theta), \tag{4}$$

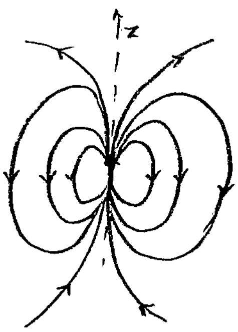
   where θ is the spherical polar coordinate and θ_{fun} stands for the Heaviside step function. Evaluate the effective charge density

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{5}$$

   for the polarized cone in terms of spherical coordinates and the respective unit vectors.

Final Exam, Prob. 1

(a)



(b)
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{d}\cdot\hat{r})\hat{r} - \vec{d}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{z^3}\left[3(d\hat{z}\cdot\hat{z})\hat{z} - d\hat{z}\right]$$

(for $\vec{d} = d\hat{z}$, $\hat{r} = \hat{z}$, $r = z$.)

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{z^3}\left[3d\hat{z} - d\hat{z}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2d}{z^3}\hat{z}$$

(on the z-axis).

Final Exam, Prob. 2

$$\int_{-1}^{1} dx\, \delta(1-3x)\left[9x^2+3x-1\right] = \int_{-1}^{1} dx\, \frac{1}{|-3|}\,\delta\left(x-\tfrac{1}{3}\right)\left[9x^2+3x-1\right]$$

$$= \frac{1}{3}\left[9\left(\tfrac{1}{3}\right)^2 + 3\left(\tfrac{1}{3}\right) - 1\right]$$

$$= \frac{1}{3}.$$

Final Exam, Prob. 3

(a) $$\int d^3r \, \rho(\vec{r}) = Q$$

$$\int_0^\infty r^2 dr \underbrace{\int d\Omega}_{4\pi} \; br \, \theta(R-r) = Q$$

$$4\pi b \int_0^R r^3 dr = Q$$

$$4\pi b \frac{R^4}{4} = Q \quad \Rightarrow \quad b = \frac{Q}{\pi R^4}$$

(b) outside: $$\vec{\nabla} \cdot \vec{E} = \rho \frac{1}{\epsilon_0}$$

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \quad \Rightarrow \quad E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

inside: $r < R$

$$4\pi r^2 E = \frac{1}{\epsilon_0} \int_0^r r'^2 dr' \, 4\pi \, br' \, \theta(R-r')$$

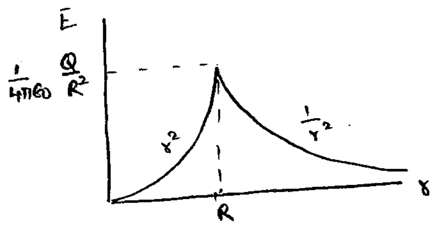
$$= \frac{1}{\epsilon_0} \cancel{4}\pi b \frac{r^4}{\cancel{4}}$$

$$= \frac{1}{\epsilon_0} Q \frac{r^4}{R^4} \quad \Rightarrow \quad E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \frac{r^4}{R^4}$$

Thus,

$$\vec{E} = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2} \dfrac{r^4}{R^4} \hat{r}, & r < R, \\[2ex] \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2} \hat{r}, & R < r. \end{cases}$$

(c)



Final Exam, prob. 4

$$\vec{P}(\vec{r}) = \vec{P}_0 \; \Theta_{fun}(\theta_0 - \theta)$$

$$\begin{aligned}
\rho_{eff}(\vec{r}) &= - \vec{\nabla} \cdot \vec{P} \\
&= - \vec{\nabla} \cdot \left[\vec{P}_0 \; \Theta_{fun}(\theta_0 - \theta) \right] \\
&= - \vec{P}_0 \cdot \vec{\nabla} \Theta_{fun}(\theta_0 - \theta) \\
&= - \vec{P}_0 \cdot (\vec{\nabla}\theta) \frac{\partial}{\partial\theta} \Theta_{fun}(\theta_0 - \theta) \\
&= \vec{P}_0 \cdot (\vec{\nabla}\theta) \; \delta(\theta - \theta_0) \\
&= \vec{P}_0 \cdot \hat{\theta} \; \frac{\delta(\theta - \theta_0)}{r}
\end{aligned}$$

$$\vec{\nabla}\theta = \frac{\hat{\theta}}{r}$$

Final Exam, prob. 5

$$2 \frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t) \quad \text{— (i)}$$

$$2 \frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t) \quad \text{— (ii)}$$

Adding (i) and (ii) we have.

$$\left(\frac{d}{dt} + \frac{m}{t} \right) J_m(t) = J_{m-1}(t).$$

Subtracting (i) from (ii) we have.

$$\left(- \frac{d}{dt} + \frac{m}{t} \right) J_m(t) = J_{m+1}(t).$$