Final Exam (Fall 2014)

PHYS 320: Electricity and Magnetism I

Date: 2014 Dec 10

1. (**20 points.**) The electric field due to a point dipole $\mathbf{d}$ at a distance $\mathbf{r}$ away from dipole is given by the expression
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}\big]. \tag{1}$$
Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{d} = d\,\hat{\mathbf{z}}$.
   (a) Qualitatively plot the electric field lines for the dipole $\mathbf{d}$. (It should be a good sketch.)
   (b) Find the (simplified) expression for the electric field on the positive z-axis. (Hint: On the positive z-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = z$.)

2. (**20 points.**) Evaluate the integral
$$\int_{-1}^{1} dx\, \delta(1-3x)\Big[9x^2 + 3x - 1\Big]. \tag{2}$$

3. (**20 points.**) Consider a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density
$$\rho(\mathbf{r}) = br\,\theta(R-r), \tag{3}$$
where r is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.
   (a) Integrating the charge density over all space gives you the total charge Q. Thus, determine the constant b in terms of Q and R.
   (b) Using Gauss's law find the electric field inside and outside the sphere.
   (c) Plot the magnitude of the electric field with respect to r.

4. (**20 points.**) Consider a right circular cone with uniform polarization $\mathbf{P}_0$, of infinite height, apex at the origin, aperture angle $2\theta_0$, described by
$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\,\theta_{\text{fun}}(\theta_0 - \theta), \tag{4}$$
where θ is the spherical polar coordinate and θ_{fun} stands for the Heaviside step function. Evaluate the effective charge density
$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{5}$$
for the polarized cone in terms of spherical coordinates and the respective unit vectors.

5. (**20 points.**) Using the recurrence relations,

$$2\frac{d}{dt}J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{6a}$$

$$2\frac{m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t), \tag{6b}$$

satisfied by the Bessel functions, derive the 'ladder' operations satisfied by the Bessel functions,

$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t), \tag{7}$$

$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t). \tag{8}$$

In quantum mechanics a ladder operator is a raising or lowering operator that transforms eigenfunctions by increasing or decreasing the eigenvalue.