Solutions

Homework No. 01 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Aug 27, 4:00 PM

Total = 210 points

10 1. Problem 1.2, Griffiths 4th edition.

10 2. Find the angle between the face diagonal (1,0,1) and the body diagonal (1,1,1) of the cube in Figure 1.10 of Griffiths 4th edition. (Modified version of Example 1.2, Griffiths 4th edition.)

10 3. Problem 1.4, Griffiths 4th edition.

10 4. Using antisymmetric property of Levi-Civita symbol show that

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = -\mathbf{A}\cdot\mathbf{C}\times\mathbf{B}. \tag{1}$$

5. In spherical polar coordinates a point is coordinated by the intersection of family of spheres, cones, and half-planes, given by

$$r = \sqrt{x^2+y^2+z^2}, \tag{2a}$$

$$\theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}, \tag{2b}$$

$$\phi = \tan^{-1}\frac{y}{x}, \tag{2c}$$

respectively. Show that the gradient of these surfaces are given by

10 — $$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{3a}$$

10 — $$\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\theta}}\frac{1}{r}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{3b}$$

10 — $$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}. \tag{3c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding
10 — normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

6. Show that

10 — (a) $\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$

10 - (b) $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$

10 - (c) $\boldsymbol{\nabla}\cdot\mathbf{r} = 3$

HW-01, prob 1

Let
$$\vec{A} = \hat{i} + \hat{j}$$
$$\vec{B} = \hat{j}$$
$$\vec{C} = \hat{k}$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = [(\hat{i} + \hat{j}) \times \hat{j}] \times \hat{k} = (\hat{i} \times \hat{j}) \times \hat{k} = 0$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\hat{i} + \hat{j}) \times (\hat{j} \times \hat{k})$$
$$= (\hat{i} + \hat{j}) \times \hat{i}$$
$$= -\hat{k}$$

Thus, $(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$

HW-01, prob 2

$$\vec{A} = 1\hat{i} + 0\hat{j} + 1\hat{k}$$
$$\vec{B} = 1\hat{i} + 1\hat{j} + 1\hat{k}$$

$A = \sqrt{2}$

$B = \sqrt{3}$

$$\vec{A} \cdot \vec{B} = 2$$

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$
$$2 = \sqrt{2}\sqrt{3}\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) = 35.26$$

HW-01, prob 3

$$\vec{A} = 1\hat{i} + 0\hat{j} + 3\hat{k}$$
$$\vec{B} = 1\hat{i} + 2\hat{j} + 0\hat{k}$$

$$\vec{A} \times \vec{B} = -6\hat{i} + 3\hat{j} + 2\hat{k}$$

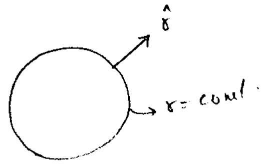
$$|\vec{A} \times \vec{B}| = \sqrt{6^2 + 3^2 + 2^2} = 7$$

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = -\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$$

HW-01, prob 4

$$\vec{A}\cdot\vec{B}\times\vec{C} = A_i (\vec{B}\times\vec{C})_i$$
$$= A_i \,\epsilon_{ijk} B_j C_k$$
$$= B_j \,\epsilon_{ijk} A_i C_k$$
$$= B_j \,\epsilon_{jki} C_k A_i = B_j (\vec{C}\times\vec{A})_j = \vec{B}\cdot\vec{C}\times\vec{A}$$
$$= A_i \,\epsilon_{jki} C_k B_j$$
$$= - A_i \,\epsilon_{ikj} C_k B_j = - A_i (\vec{C}\times\vec{B})_i = -\vec{A}\cdot\vec{C}\times\vec{B}$$

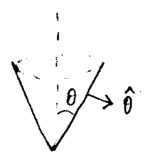
HW-01, prob 5

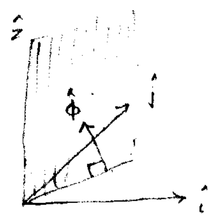


$$\vec{\nabla} r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$
$$= \hat{i}\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{j}\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$
$$\sec^2\theta \frac{d\theta}{dx} = \frac{x}{z\sqrt{x^2+y^2}}$$
$$\Rightarrow \frac{d\theta}{dx} = \frac{x\cos^2\theta}{z\sqrt{x^2+y^2}}$$
$$= \frac{r\sin\theta\cos\phi\cos^2\theta}{r\cos\theta \; r\sin\theta}$$
$$= \frac{1}{r}\cos\theta\cos\phi$$

$$\vec{\nabla}\theta = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$
$$= \frac{1}{\sec^2\theta}\left(\hat{i}\frac{x}{z\sqrt{x^2+y^2}} + \hat{j}\frac{y}{z\sqrt{x^2+y^2}} - \hat{k}\frac{\sqrt{x^2+y^2}}{z^2}\right)$$
$$= \frac{1}{r}\cos\theta\cos\phi\,\hat{i} + \frac{1}{r}\cos\theta\sin\phi\,\hat{j} - \frac{\sin\theta}{r}\hat{k}$$





$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\tan^{-1}\left(\frac{y}{x}\right)$$
$$= \frac{1}{\sec^2\phi}\left(-\frac{y}{x^2}\hat{i} + \frac{1}{x}\hat{j} + 0\hat{k}\right)$$
$$= -\frac{\sin\phi}{r\sin\theta}\hat{i} + \frac{\cos\phi}{r\sin\theta}\hat{j} + 0\hat{k}$$

HW-01, prob 6

(a) $$\vec{\nabla} r = \vec{\nabla}\sqrt{x^2+y^2+z^2}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \hat{i}\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{j}\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}$$

$$= \frac{\vec{r}}{r}$$

$$= \hat{r}$$

(b) $$\vec{\nabla}\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\frac{\partial}{\partial x}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$+ \hat{j}\frac{\partial}{\partial y}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$+ \hat{k}\frac{\partial}{\partial z}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$$

(c) $$\vec{\nabla}\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{j} + \hat{k}\cdot\hat{k}$$

$$= 3$$

(d) $$\vec{\nabla}\times\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}\times\hat{i} + \hat{j}\times\hat{j} + \hat{k}\times\hat{k}$$

$$= 0$$

HW-01, prob 7

(a) $\vec{\nabla}\frac{1}{r^n} = -\frac{n}{r^{n+1}}\vec{\nabla} r$ (using $\vec{\nabla} r = \hat{r}$)

$= -\frac{n}{r^{n+1}}\hat{r} = -n\frac{\vec{r}}{r^{n+2}}$

(b) $\vec{\nabla}\left(\frac{\vec{r}}{r^n}\right) = (\vec{\nabla}\vec{r})\frac{1}{r^n} + \vec{r}\left(\vec{\nabla}\frac{1}{r^n}\right)$

$= \overleftrightarrow{1}\frac{1}{r^n} + \vec{r}\left(\frac{-n}{r^{n+2}}\vec{r}\right)$ (using $\vec{\nabla}\vec{r} = \overleftrightarrow{1}$, $\vec{\nabla}\frac{1}{r^n} = -n\frac{\vec{r}}{r^{n+2}}$)

$= \overleftrightarrow{1}\frac{1}{r^n} - \vec{r}\vec{r}\frac{n}{r^{n+2}}$

(c) $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^n}\right) = (\vec{\nabla}\cdot\vec{r})\frac{1}{r^n} + \vec{r}\cdot\left(\vec{\nabla}\frac{1}{r^n}\right)$

$= 3\frac{1}{r^n} + \vec{r}\cdot\left(\frac{-n}{r^{n+2}}\vec{r}\right)$ ($\vec{r}\cdot\vec{r} = r^2$.)

$= \frac{3}{r^n} - \frac{n}{r^{n+2}}r^2$

$= \frac{3-n}{r^n}$

(d) $\vec{\nabla}\times\left(\frac{\vec{r}}{r^n}\right) = (\vec{\nabla}\times\vec{r})\frac{1}{r^n} + \vec{r}\times\left(\vec{\nabla}\frac{1}{r^n}\right)$

$= 0 - \frac{n}{r^{n+2}}\vec{r}\times\vec{r}$

$= 0$

(e) $\vec{\nabla}\cdot\vec{v} = \left.\frac{3-n}{r^n}\right|_{n=3} = 0$

This is true for $r \neq 0$. For $r=0$ it tends to ∞.

In particular, $\vec{\nabla}\cdot\vec{v} = 4\pi\,\delta(x)\,\delta(y)\,\delta(z)$.

HW-01, prob 8

(a)

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})]_i = \epsilon_{ijk}\nabla_j\epsilon_{kmn}\nabla_m A_n$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\nabla_j\nabla_m A_n.$$

$$= \nabla_j\nabla_i A_j - \nabla_j\nabla_j A_i$$

$$= \nabla_i \nabla_j A_j - \nabla_j\nabla_j A_i$$

$$= \nabla_i\, \vec{\nabla}\cdot\vec{A} - \nabla^2 A_i$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

(b)

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \nabla_i(\vec{A}\times\vec{B})_i$$

$$= \nabla_i(\epsilon_{ijk}A_j B_k)$$

$$= \epsilon_{ijk}\nabla_i(A_j B_k)$$

$$= \epsilon_{ijk}(\nabla_i A_j) B_k + \epsilon_{ijk} A_j (\nabla_i B_k)$$

$$= (\epsilon_{kij}\nabla_i A_j) B_k - A_j(\epsilon_{jik}\nabla_i B_k)$$

$$= (\vec{\nabla}\times\vec{A})_k B_k - A_j(\vec{\nabla}\times\vec{B})_j$$

$$= (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$