Solutions

Homework No. 02 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Monday, 2014 Sep 8, 4:00 PM

1. (**10 points.**) A gyroid, see Fig. 1, is an (infinitely connected triply periodic minimal) surface discovered by Alan Schoen in 1970. Schoen presently resides in Carbondale and was a professor at SIU in the later part of his career. Apparently, a gyroid is approximately described by the surface

$$f(x, y, z) = \cos x \sin y + \cos y \sin z + \cos z \sin x \tag{1}$$

when $f(x, y, z) = 0$. Using the fact that the gradient operator

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{2}$$

determines the normal vectors on a surface, evaluate

$$\boldsymbol{\nabla} f(x, y, z). \tag{3}$$

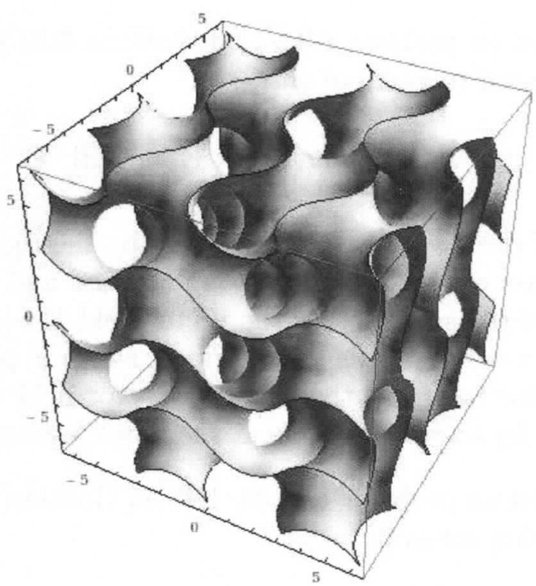


Figure 1: Problem 1.

(d) $\int_{-2}^{2} dx\, [3x+3]\, \delta(3x)$

(e) $\int_{-2}^{2} dx\, [3x+3]\, \delta(-3x)$

(f) $\int_{0}^{2} dx\, [3x+3]\, \delta(1-x)$

(g) $\int_{-1}^{1} dx\, 9x^3\, \delta(3x+1)$

7. (**30 points.**) (Based on problem 1.47/1.46 Griffiths 4th/3rd edition.)

   (a) Express the charge density $\rho(\mathbf{r})$ of a point charge Q positioned at $\mathbf{r}_a$ in terms of δ-functions. Verify that the volume integral of ρ equals Q.

   (b) Express the charge density of an infinitely long wire, of uniform charge per unit length λ and parallel to z-axis, in terms of δ-functions.

   (c) Express the charge density of an infinite plate, of uniform charge per unit area σ and parallel to xy-plane, in terms of δ-functions.

HW-02, prob 1

$$f(x, y, z) = \cos x \sin y + \cos y \sin z + \cos z \sin x.$$

$$\vec{\nabla} f = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) f$$

$$= \left(\cos z \cos x - \sin x \sin y \right) \hat{i} + \left(\cos x \cos y - \sin y \sin z \right) \hat{j}$$

$$+ \left(\cos y \cos z - \sin z \sin x \right) \hat{k}$$

HW-02, prob. 2

(a)
$$\nabla^2 T_a = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(x^2 + 2xy + 3z + 4 \right)$$

$$= 2 + 0 + 0 = 2.$$

(b)
$$\nabla^2 T_b = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \sin x \sin y \sin z$$

$$= - \sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= - 3 \sin x \sin y \sin z$$

(c)
$$\nabla^2 \vec{V} = \left(\nabla^2 x^2 \right) \hat{x} + \left(\nabla^2 3xz^2 \right) \hat{y} - \left(\nabla^2 2xz \right) \hat{z}$$

$$= 2 \hat{x} + 6x \hat{y} + 0 \hat{z}$$

HW-02, prob 3

$$T(\vec{x}) = x^2 + 4xy + 2yz^3$$

$$\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k} \qquad T(\vec{a}) = 0$$

$$\vec{b} = \hat{i} + \hat{j} + \hat{k} \qquad T(\vec{b}) = 7$$

$$T(\vec{b}) - T(\vec{a}) = 7$$

$$\vec{\nabla} T = \hat{i}\frac{\partial}{\partial x}T + \hat{j}\frac{\partial}{\partial y}T + \hat{k}\frac{\partial}{\partial z}T$$

$$= (2x+4y)\hat{i} + (4x+2z^3)\hat{j} + (6yz^2)\hat{k}$$

Path 1 : $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$$\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{\nabla}T = \int_{x=0}^{x=1} dx\,\hat{i}\cdot\vec{\nabla}T\Big|_{\substack{y=0\\z=0}} + \int_{y=0}^{y=1} dy\,\hat{j}\cdot\vec{\nabla}T\Big|_{\substack{x=1\\z=0}} + \int_{z=0}^{z=1} dz\,\hat{k}\cdot\vec{\nabla}T\Big|_{\substack{x=1\\y=1}}$$

$$= \int_{x=0}^{x=1} dx\,(2x+4y)\Big|_{\substack{y=0\\z=0}} + \int_{y=0}^{y=1} dy\,(4x+2z^3)\Big|_{\substack{x=1\\z=0}} + \int_{z=0}^{z=1} dz\,6yz^2\Big|_{\substack{x=1\\y=1}}$$

$$= 1 + 4 + 2 = 7 \quad ✓$$

Path 2 : $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

$$\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{\nabla}T = \int_{z=0}^{z=1} dz\,\hat{k}\cdot\vec{\nabla}T\Big|_{\substack{x=0\\y=0}} + \int_{y=0}^{y=1} dy\,\hat{j}\cdot\vec{\nabla}T\Big|_{\substack{x=0\\z=1}} + \int_{x=0}^{x=1} dx\,\hat{i}\cdot\vec{\nabla}T\Big|_{\substack{y=1\\z=1}}$$

$$= \int_{z=0}^{z=1} dz\,6yz^2\Big|_{\substack{x=0\\y=0}} + \int_{y=0}^{y=1} dy\,(4x+2z^3)\Big|_{\substack{x=0\\z=1}} + \int_{x=0}^{x=1} dx\,(2x+4y)\Big|_{\substack{y=1\\z=1}}$$

$$= 0 + 2 + 5$$

$$= 7 \quad ✓$$

Path 3: along $z = x^2$, $y = x$.

$z = x^2 \Rightarrow dz = 2x\,dx$.

$y = x \Rightarrow dy = dx$

Thus, $d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$= dx\,\hat{i} + dx\,\hat{j} + 2x\,dx\,\hat{k}$

$= dx\,(\hat{i} + \hat{j} + 2x\,\hat{k})$

$$\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{\nabla}T = \int_{x=0}^{x=1} dx\,(\hat{i}+\hat{j}+2x\,\hat{k})\cdot\vec{\nabla}T\Big|_{\substack{y=x\\ z=x^2}}$$

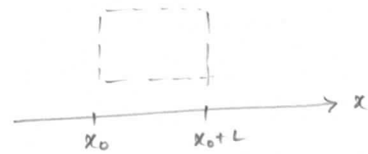
$$= \int_{x=0}^{x=1} dx\,(\hat{i}+\hat{j}+2x\,\hat{k})\cdot\left(6x\,\hat{i} + [4x+2x^6]\,\hat{j} + 6x^5\,\hat{k}\right)$$

$$= \int_{x=0}^{x=1} dx\,\left[6x + 4x + 2x^6 + 12x^6\right]$$

$$= \int_{x=0}^{x=1} dx\,\left[10x + 14x^6\right]$$

$= 5 + 2 = 7.$ ✓

HW-02, prob. 4



$$\vec{E} = x\,\hat{i}$$

$$\vec{\nabla}\cdot\vec{E} = 1$$

$$\int_V d^3x\ \vec{\nabla}\cdot\vec{E} = \int_V d^3x = L^3$$

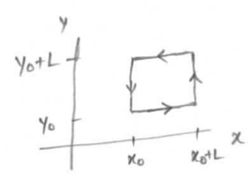
$$\oint_S d\vec{a}\cdot\vec{E} = \underbrace{\int dz\int dy}_{L^2}\ (-\hat{i})\cdot x\,\hat{i}\Big|_{x=x_0} + \underbrace{\int dz\int dy}_{L^2}\ \hat{i}\cdot x\,\hat{i}\Big|_{x=x_0+L}$$

$$= L^2\left(-x_0 + x_0 + L\right)$$

$$= L^3.$$

Distribution of source/sink $\rightarrow$ $\vec{\nabla}\cdot\vec{E} = 1$ (uniform)

HW-02, prob. 5



$$\vec{E} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$$

$$\vec{\nabla}\times\vec{E} = -\hat{i} - \hat{j} - \hat{k}$$

$$\int_S d\vec{a}\cdot\vec{\nabla}\times\vec{E} = \int_{x_0}^{x_0+L} dx\int_{y_0}^{y_0+L} dy\ \hat{k}\cdot(-\hat{i}-\hat{j}-\hat{k}) = -L^2$$

$$\oint_C d\vec{l}\cdot\vec{E} = \int_{x=x_0}^{x_0+L} dx\ \hat{i}\cdot(y\,\hat{i}+z\,\hat{j}+x\,\hat{k})\Big|_{\substack{z=0\\ y=y_0}} + \int_{y=y_0}^{y=y_0+L} dy\ \hat{j}\cdot(y\,\hat{i}+z\,\hat{j}+x\,\hat{k})\Big|_{\substack{z=0\\ x=x_0+L}}$$

$$+\int_{x_0}^{x_0+L} dx\,(-\hat{i})\cdot(y\,\hat{i}+z\,\hat{j}+x\,\hat{k})\Big|_{\substack{z=0\\ y=y_0+L}} + \int_{y_0}^{y_0+L} dy\,(-\hat{j})\cdot(y\,\hat{i}+z\,\hat{j}+x\,\hat{k})\Big|_{\substack{z=0\\ x=x_0}}$$

$$= y_0\int_{x=x_0}^{x_0+L} dx + 0 - (y_0+L)\int_{x_0}^{x_0+L} dx + 0$$

$$= -L^2$$

Torque field $\longrightarrow$ $\vec{\nabla}\times\vec{E} = -\hat{i} - \hat{j} - \hat{k}$.

HW-02, prob. 6

(a) $$\int_2^6 dx\,[3x^2-2x-3]\,\delta(x-3) = 3\times3^2-2\times3-3 = 27-6-3 = 18$$

(b) $$\int_{-7}^{7} dx\,\sin x\,\delta(x-\pi) = \sin\pi = 0$$

(c) $$\int_0^3 dx\,x^3\,\delta(x+1) = 0$$ ($\because$ $x=-1$ is not in integral range.)

(d) $$\int_{-2}^{2} dx\,[3x+3]\,\delta(3x) = \int_{-2}^{2} dx\,[3x+3]\,\frac{1}{3}\,\delta(x) = \frac{1}{3}[3\times0+3] = 1$$

$$\delta(-3x) = \frac{1}{|-3|}\delta(x) = \frac{1}{3}\delta(x)$$

(e) $$\int_{-2}^{2} dx\,[3x+3]\,\delta(-3x) = \int_{-2}^{2} dx\,[3x+3]\,\frac{1}{3}\,\delta(x) = \frac{1}{3}[3\times0+3] = 1$$

(f) $$\int_0^2 dx\,[3x+3]\,\delta(1-x) = \int_0^2 dx\,[3x+3]\,\frac{1}{|-1|}\,\delta(x-1) = 3\times1+3 = 6$$

(g) $$\int_{-1}^{1} dx\,9x^3\,\delta(3x+1) = \int_{-1}^{1} dx\,9x^3\,\frac{1}{3}\,\delta\left(x+\frac{1}{3}\right) = 9\times\left(-\frac{1}{3}\right)^3\times\frac{1}{3} = -\frac{1}{9}$$

HW-02, prob 7

(a) $$\rho(\vec{r}) = Q\,\delta^{(3)}(\vec{r}-\vec{r}_a)$$

$$= Q\,\delta(x-x_a)\,\delta(y-y_a)\,\delta(z-z_a)$$

$$\int d^3r\,\rho(\vec{r}) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\; Q\,\delta(x-x_a)\,\delta(y-y_a)\,\delta(z-z_a)$$

$$= Q \underbrace{\int_{-\infty}^{+\infty} dx\,\delta(x-x_a)}_{=1} \underbrace{\int_{-\infty}^{+\infty} dy\,\delta(y-y_a)}_{=1} \underbrace{\int_{-\infty}^{+\infty} dz\,\delta(z-z_a)}_{=1}$$

$$= Q$$

(b) $$\rho(\vec{r}) = \lambda\,\delta(x-x_0)\,\delta(y-y_0)$$

(c) $$\rho(\vec{r}) = \sigma\,\delta(z-z_0)$$