Homework No. 05 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Oct 1, 4:00 PM

Solutions

1. **(40 points.)** Consider an electric dipole at the origin, with the negative charge at the coordinate $(0, 0, -a)$ and the positive charge at $(0, 0, a)$, such that the electric dipole moment $\mathbf{p}$ points along the z-axis, $p = 2aq$.

   (a) Show that the electric potential due to the dipole at the point

   $$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{1}$$

   is given by the expression

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}}\right]. \tag{2}$$

   (b) For $a \ll r$ show that the potential is approximately given by

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{pz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \tag{3}$$

   (c) Consider the limit when a is made to vanish while q becomes infinite, in such a way that $2aq$ remains the finite value p. This is a point dipole. The electric potential for a point dipole is exactly described by Eq. (3). Using polar coordinates write $z = r\cos\theta$ and thus rewrite the potential of a point dipole in Eq. (3) in the form

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{4}$$

   (d) Evaluate the electric field due to a point dipole using

   $$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{5}$$

   and express it in the following form,

   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\big]. \tag{6}$$

   Draw the electric field lines of a point dipole for $\mathbf{p} = p\,\hat{\mathbf{z}}$.

HW-05, prob 1

$\vec{r} = (x, y, z)$

$\vec{r}_1 = (0, 0, a)$

$\vec{r}_2 = (0, 0, -a)$

(a) $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{(+q)}{|\vec{r}-\vec{r}_1|} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{|\vec{r}-\vec{r}_2|}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+a)^2}}\right]$$

(b) For $a \ll r$ we have.

$$\frac{1}{\sqrt{1\pm x}} \approx 1 \mp \frac{1}{2}x$$

$$\frac{q}{\sqrt{x^2+y^2+(z\pm a)^2}} = \frac{q}{\sqrt{r^2 \pm 2az + a^2}}$$

$$= \frac{q}{r}\frac{1}{\sqrt{1 \pm \left(\frac{2az}{r^2} \pm \frac{a^2}{r^2}\right)}}$$

$$\approx \frac{q}{r}\frac{1}{\sqrt{1 \pm \frac{2az}{r^2}}}$$ because $\left(\frac{a}{r}\right)^2 \ll \left(\frac{a}{r}\right) \ll 1$

$$\approx \frac{q}{r}\left[1 \mp \frac{1}{2}\frac{2az}{r^2}\right]$$

Thus,

$$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r}\left(1+\frac{az}{r^2}\right) - \frac{q}{r}\left(1-\frac{az}{r^2}\right)\right]$$

$2aq = p$

$$= \frac{1}{4\pi\epsilon_0}\frac{2aq\,z}{r^3}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{pz}{r^3}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{pz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

(c)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{pz}{r^3} \qquad z = r\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{r^2} \qquad \vec{p}\cdot\vec{r} = pr\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\vec{r}}{r^3}$$

(d)

$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\frac{1}{4\pi\epsilon_0} \vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left(\vec{\nabla}\,\vec{p}\cdot\vec{r}\right)\frac{1}{r^3} - \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\vec{r}\;\vec{\nabla}\frac{1}{r^3}$$

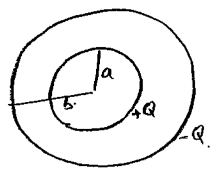
$$= -\frac{1}{4\pi\epsilon_0}\vec{p}\,\frac{1}{r^3} - \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\vec{r}\left(-3\frac{\vec{r}}{r^5}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p}\right]$$

$$\vec{\nabla}\,\vec{p}\cdot\vec{r} = \vec{\nabla}\,\vec{r}\cdot\vec{p} = \overleftrightarrow{1}\cdot\vec{p} = \vec{p}$$

$$\vec{\nabla}\frac{1}{r^3} = -\frac{3}{r^4}\vec{\nabla}r = -3\frac{\vec{r}}{r^5}$$

prob 2, HW-05



$$\vec{E} = \begin{cases} 0 & 0 \le r < a \\ \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} & a < r < b \\ 0 & b < r < \infty \end{cases}$$

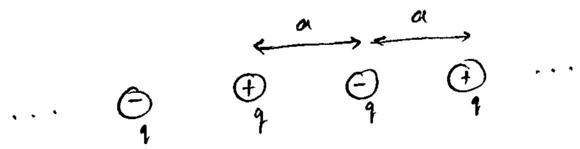
$$V = -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot d\vec{r}$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Comparing with $|V| = \frac{Q}{C}$

$$C = \frac{4\pi\epsilon_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} \xrightarrow{b\to\infty} 4\pi\epsilon_0 a.$$

HW-05, prob 3



$$\left.\begin{array}{l}\text{Potential}\\\text{per charge}\end{array}\right\} = -2\frac{1}{4\pi\epsilon_0}\frac{q}{a} + 2\frac{1}{4\pi\epsilon_0}\frac{q}{2a} - 2\frac{1}{4\pi\epsilon_0}\frac{q}{3a} + \cdots$$

$$= -2\frac{1}{4\pi\epsilon_0}\frac{q}{a}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{a}\,2\ln 2$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{a}M \qquad M = -2\ln 2$$