Homework No. 05 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Oct 1, 4:00 PM

1. (**40 points.**) Consider an electric dipole at the origin, with the negative charge at the coordinate $(0, 0, -a)$ and the positive charge at $(0, 0, a)$, such that the electric dipole moment $\mathbf{p}$ points along the z-axis, $p = 2aq$.

   (a) Show that the electric potential due to the dipole at the point

   $$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{1}$$

   is given by the expression

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}}\right]. \tag{2}$$

   (b) For $a \ll r$ show that the potential is approximately given by

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{pz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \tag{3}$$

   (c) Consider the limit when a is made to vanish while q becomes infinite, in such a way that $2aq$ remains the finite value p. This is a point dipole. The electric potential for a point dipole is exactly described by Eq. (3). Using polar coordinates write $z = r\cos\theta$ and thus rewrite the potential of a point dipole in Eq. (3) in the form

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{4}$$

   (d) Evaluate the electric field due to a point dipole using

   $$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{5}$$

   and express it in the following form,

   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\big]. \tag{6}$$

   Draw the electric field lines of a point dipole for $\mathbf{p} = p\,\hat{\mathbf{z}}$.

2. (**20 points.**) Determine the capacitance of a spherical capacitor, consisting of concentric spheres of radius a and b, $b > a$, to be

$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}. \tag{7}$$

Take the limit $b \to \infty$ to determine the so-called self-capacitance of an isolated conducting sphere.

3. (**20 points.**) Consider an infinite chain of equidistant alternating point charges $+q$ and $-q$ on the x-axis. Calculate the electric potential at the site of a point charge due to all other charges. This is equal to the work per point charge required to assemble such a cofiguration. In terms of the distane a between neighbouring charges we can derive an expression for this energy to be

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{a}M, \tag{8}$$

where M is defined as the Madelung constant for this hypothetical one-dimensional crystal. Determine M as an infinite sum, and evaluate the sum. (Madelung contants for three-dimensional crystals involve triple sums, which are typically a challenge to evaluate because of slow convergence.)