Solution

Homework No. 06 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Friday, 2014 Oct 10, 4:00 PM

1. (**30 points.**) The relation between charge density and current density,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{1}$$

is the general statement of the conservation of charge.

   (a) Derive the statement of conservation of charge in Eq. (1) from the Maxwell equations. Hint: Take time derivative of Gauss's law and divergence of Ampere's law.

   (b) For an arbitrarily moving point particle with charge, the charge and current densities are

$$\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)) \tag{2}$$

   and

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}_a(t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)), \tag{3}$$

   where $\mathbf{r}_a(t)$ is the position vector and

$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{4}$$

   is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (1) for a point particle.

2. (**20 points.**) (Ref. Schwinger et al., problem 4.1.)
Consider the charge density

$$\rho(\mathbf{r}) = -\mathbf{d}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}). \tag{5}$$

   (a) Find the total charge of the charge density by evaluating

$$\int d^3r\,\rho(\mathbf{r}). \tag{6}$$

   Hint: Integrate by parts.

   (b) Find the dipole moment of the charge density by evaluating

$$\int d^3r\,\mathbf{r}\,\rho(\mathbf{r}). \tag{7}$$

   Hint: Integrate by parts, and use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$.

HW-06, prob 1

(a) $$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho$$

$$\Rightarrow \frac{\partial}{\partial t} \vec{\nabla} \cdot \epsilon_0 \vec{E} = \frac{\partial \rho}{\partial t} \qquad \text{— (i)}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t} \epsilon_0 \vec{E} + \vec{J}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{\nabla} \times \vec{H} = \vec{\nabla} \cdot \frac{\partial}{\partial t} \epsilon_0 \vec{E} + \vec{\nabla} \cdot \vec{J} \qquad \text{— (ii)}$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{H} = 0$$

$$\frac{\partial}{\partial t} \vec{\nabla} \cdot \epsilon_0 \vec{E} = \vec{\nabla} \cdot \frac{\partial}{\partial t} \epsilon_0 \vec{E}$$

Adding (i) and (ii) we have.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

(b) For a point charge.

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} q \, \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$= q \frac{\partial \vec{r}_a(t)}{\partial t} \cdot \frac{\partial}{\partial \vec{r}_a(t)} \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

(using chain rule)

$$= -q \, \vec{V}_a(t) \cdot \vec{\nabla} \, \delta^{(3)}(\vec{r} - \vec{r}_a(t))$$

$$\left(\frac{\partial}{\partial x} f(x-y) = -\frac{\partial}{\partial y} f(x-y) \right)$$

$$= -\vec{\nabla} \cdot \left[q \, \vec{V}_a(t) \, \delta^{(3)}(\vec{r} - \vec{r}_a(t)) \right]$$

($\vec{V}_a(t)$ is independent of position.)

$$= -\vec{\nabla} \cdot \vec{J}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

HW-06, prob 2

$$\rho(\vec{r}) = -\vec{d}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

(a)
$$\int d^3r\,\rho(\vec{r}) = \int d^3r\,(-\vec{d}\cdot\vec{\nabla})\,\delta^{(3)}(\vec{r})$$

($\vec{d}$ is uniform)

$$= -\vec{d}\cdot\int d^3r\,\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

(because it is a total derivative.)

$$= 0$$

The total charge of a dipole is indeed zero.

(b)
$$\int d^3r\,\vec{r}\,\rho(\vec{r}) = \int d^3r\,\vec{r}\,(-\vec{d}\cdot\vec{\nabla})\,\delta^{(3)}(\vec{r})$$

$$= \int d^3r\,(-\vec{d}\cdot\vec{\nabla})\left[\vec{r}\,\delta^{(3)}(\vec{r})\right] - \int d^3r\,\underbrace{\left[(-\vec{d}\cdot\vec{\nabla})\,\vec{r}\right]}_{=-\vec{d}\cdot\overleftrightarrow{1}}\,\delta^{(3)}(\vec{r})$$

$= 0$ at the surface due to δ-function.

$$= \vec{d}\cdot\overleftrightarrow{1}\,\underbrace{\int d^3r\,\delta^{(3)}(\vec{r})}_{=1}$$

$$= \vec{d}$$

HW-06, prob 3

$$\rho(\vec{r}) = \vec{\nabla}\cdot\epsilon_0\vec{E}$$

$$= \vec{\nabla}\cdot\left[-\alpha\,\vec{r}\,\theta(R-r)\right]$$

$$= -\alpha\,\underbrace{(\vec{\nabla}\cdot\vec{r})}_{=3}\,\theta(R-r) - \alpha\,\vec{r}\cdot\vec{\nabla}\,\theta(R-r)$$

$$= -3\alpha\,\theta(R-r) - \alpha\,\vec{r}\cdot\underbrace{(\vec{\nabla}r)}_{\hat{r}}\,\underbrace{\frac{\partial}{\partial r}\theta(R-r)}_{-\delta(R-r)}$$

$$= -3\alpha\,\theta(R-r) + \alpha r\,\delta(R-r)$$

$$= -3\alpha\,\theta(R-r) + \alpha R\,\delta(R-r)$$

$$=$$

HW-06, prob 4

(a) $$\int d^3r\, \rho(\vec{r}) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \;\; \lambda\, \delta(x)\, \delta(y)\; \theta(-L<z<L)$$

$$= \lambda \int_{-L}^{L} dz$$

$$= \lambda\, 2L \quad = \text{total charge on the rod.}$$

(b) $$-\nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0}\, \rho(\vec{r})$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \;\; \frac{\lambda\, \delta(x')\, \delta(y')\; \theta(-L<z'<L)}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$z - z' = \rho \sinh\theta$

$-dz' = \rho \cosh\theta \, d\theta$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} dz' \, \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_{\sinh^{-1}\left(\frac{z-L}{\rho}\right)}^{\sinh^{-1}\left(\frac{z+L}{\rho}\right)} \frac{\cancel{\rho \cosh\theta}\, d\theta}{\cancel{\rho \cosh\theta}}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\sinh^{-1}\left(\frac{z+L}{\rho}\right) - \sinh^{-1}\left(\frac{z-L}{\rho}\right) \right]$$

$\rho = \sqrt{x^2 + y^2}$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\sinh^{-1}\left(\frac{z+L}{\rho}\right) + \sinh^{-1}\left(\frac{L-z}{\rho}\right) \right]$$

(c) Let $y = \sinh^{-1} t$

$$\Rightarrow \quad t = \sinh y = \frac{e^{y} - e^{-y}}{2}$$

$$\Rightarrow \quad (e^{y})^{2} - 2t\,e^{y} - 1 = 0$$

$$e^{y} = \frac{2t \pm \sqrt{4t^{2} + 4}}{2}$$

$$e^{y} = t \pm \sqrt{t^{2} + 1}$$

$$\sinh^{-1} t = y = \ln\left(t \pm \sqrt{t^{2}+1}\right)$$

(d) Using (c) in (b)

$$\phi(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left\{\frac{L+z}{\rho} + \sqrt{\left(\frac{L+z}{\rho}\right)^{2} + 1}\right\} + \ln\left\{\frac{L-z}{\rho} + \sqrt{\left(\frac{L-z}{\rho}\right)^{2} + 1}\right\}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left\{\frac{L}{\rho}\left[1 + \frac{z}{L} + \sqrt{\left(1 + \frac{z}{L}\right)^{2} + \frac{\rho^{2}}{L^{2}}}\right]\right\} + \ln\left\{\frac{L}{\rho}\left[1 - \frac{z}{L} + \sqrt{\left(1 - \frac{z}{L}\right)^{2} + \frac{\rho^{2}}{L^{2}}}\right]\right\}\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\frac{L}{\rho} + \ln\frac{L}{\rho} + F\left(\frac{z}{L}, \frac{\rho}{L}\right)\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L}, \frac{\rho}{L}\right)\right]$$

where

$$F(a,b) = \ln\left[1 - a + \sqrt{(1-a)^{2} + b^{2}}\right] + \ln\left[1 + a + \sqrt{(1+a)^{2} + b^{2}}\right]$$

(e) Series expanding $F(a,b)$

$$F(a,b) = F(0,0) + O(a)\,O(b)$$

$$= 2\ln 2 + \underbrace{O(a,b)}_{\text{order}}.$$

Thus,

$$\phi(\vec{r}) \approx \frac{\lambda}{4\pi\epsilon_0}\left[-2\ln\frac{\rho}{L} + 2\ln 2 + \text{higher orders}\right]$$

$$\approx -\frac{2\lambda}{4\pi\epsilon_0}\ln\frac{\rho}{2L}$$

(f)

$$\vec{E} = -\vec{\nabla}\phi$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\vec{\nabla}\ln\frac{\rho}{2L}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\vec{\nabla}\left(\ln\rho - \underbrace{\ln 2L}_{\text{constant}}\right)$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\vec{\nabla}\ln\rho$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\frac{1}{\rho}\vec{\nabla}\rho$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\frac{\hat{\rho}}{\rho}$$

$$\vec{\nabla}\rho = \vec{\nabla}\sqrt{x^2+y^2}$$

$$= \hat{i}\,\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} + \hat{j}\,\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{\vec{\rho}}{\rho} = \hat{\rho}$$