Homework No. 06 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Friday, 2014 Oct 10, 4:00 PM

1. (**30 points.**) The relation between charge density and current density,

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0, \tag{1}$$

is the general statement of the conservation of charge.

   (a) Derive the statement of conservation of charge in Eq. (1) from the Maxwell equations. Hint: Take time derivative of Gauss's law and divergence of Ampere's law.

   (b) For an arbitrarily moving point particle with charge, the charge and current densities are

$$\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)) \tag{2}$$

   and

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}_a(t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)), \tag{3}$$

   where $\mathbf{r}_a(t)$ is the position vector and

$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{4}$$

   is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (1) for a point particle.

2. (**20 points.**) (Ref. Schwinger et al., problem 4.1.)
Consider the charge density

$$\rho(\mathbf{r}) = -\mathbf{d}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}), \tag{5}$$

where $\mathbf{d}$ is constant (uniform in space).

   (a) Find the total charge of the charge density by evaluating

$$\int d^3r\,\rho(\mathbf{r}). \tag{6}$$

   Hint: Use theorem of gradient.

   (b) Find the dipole moment of the charge density by evaluating

$$\int d^3r\,\mathbf{r}\,\rho(\mathbf{r}). \tag{7}$$

   Hint: Integrate by parts, and use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$.

3. (**10 points.**) (Motivated from problem 2.46 Griffiths 4th edition.)
If the electric field is given (in spherical coordinates) by the expression

$$\mathbf{E}(\mathbf{r}) = -\frac{\alpha}{\varepsilon_0}\,\mathbf{r}\,\theta(R-r), \tag{8}$$

for constant α, show that the charge density is

$$\rho(\mathbf{r}) = -3\alpha\theta(R-r) + \alpha r\delta(r-R), \tag{9}$$

where $\theta(x)$ is the Heaviside step function and $\delta(x)$ is the Dirac delta function.

4. (**60 points.**) Consider a line segment of length $2L$ with uniform line charge density λ.

   (a) When the rod is placed on the z-axis centered on the origin, show that the charge density can be expressed as

   $$\rho(\mathbf{r}) = \lambda\delta(x)\delta(y)\theta(-L<z<L), \tag{10}$$

   where $\theta(-L<z<L)=1$, if $-L<z<L$ and $\theta(z)=0$, otherwise.

   (b) Inverting the Poisson equation for the electric potential, using the Green's function, evaluate the electric potential for the rod as

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right)\right]. \tag{11}$$

   (c) Using $\sinh t = (e^t - e^{-t})/2$, show that

   $$\sinh^{-1} t = \ln(t+\sqrt{t^2+1}). \tag{12}$$

   (d) Thus, express the electric potential of Eq. (11) in the form

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L},\frac{\rho}{L}\right)\right], \tag{13}$$

   where $\rho^2 = x^2+y^2$ and

   $$F(a,b) = \ln[1-a+\sqrt{(1-a)^2+b^2}] + \ln[1+a+\sqrt{(1+a)^2+b^2}]. \tag{14}$$

   (e) An infinite rod (on the z axis) is obtained by taking the limit $\rho \ll L, z \ll L$. Show that

   $$\phi(\mathbf{r}) \xrightarrow{\rho\ll L, z\ll L} -\frac{2\lambda}{4\pi\varepsilon_0}\ln\frac{\rho}{2L}. \tag{15}$$

   Hint: Series expand and keep only leading order terms.

   (f) Using $\mathbf{E} = -\boldsymbol{\nabla}\phi$ determine the electric field for an infinite rod (placed on the z-axis) to be

   $$\mathbf{E}(\mathbf{r}) = \frac{2\lambda}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}. \tag{16}$$