Solutions

Homework No. 07 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Monday, 2014 Oct 27, 4:00 PM

1. **(30 points.)** (Based on Griffiths 3rd/4th ed., Problem 4.9.)
   (a) The electric field of a point charge q at distance $\mathbf{r}$ is
   $$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}. \tag{1}$$
   The force on a point dipole in the presence of an electric field is
   $$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{2}$$
   Use these to find the force on a point dipole due to a point charge.
   (b) The electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by Eq. (4). The force on a point charge in the presence of an electric field is
   $$\mathbf{F} = q\mathbf{E}. \tag{3}$$
   Use these to find the force on a point charge due to a point dipole.
   (c) Confirm that above two forces are equal in magnitude and opposite in direction, as per Newton's third law.

2. **(40 points.)** (Based on Griffiths 3rd/4th ed., Problem 4.8.)
   We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression
   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{4}$$
   The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by
   $$U = -\mathbf{d}\cdot\mathbf{E}. \tag{5}$$
   Further, the force between the two dipoles is given by
   $$\mathbf{F} = -\boldsymbol{\nabla}U. \tag{6}$$
   Use these expressions to derive

HW-07, prob 1

(a) $$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

$$\vec{\nabla}\vec{r} = \overleftrightarrow{1}$$
$$\vec{\nabla} r = \hat{r}$$

$$\begin{aligned}\vec{F}_{dipole} &= \vec{d}\cdot\vec{\nabla}\,\vec{E} \\ &= \vec{d}\cdot\vec{\nabla}\,\frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \\ &= \frac{q}{4\pi\epsilon_0}\,\vec{d}\cdot\vec{\nabla}\,\frac{\vec{r}}{r^3} \\ &= \frac{q}{4\pi\epsilon_0}\,\vec{d}\cdot\left[\frac{(\vec{\nabla}\vec{r})}{r^3} - 3\frac{\vec{r}\,\vec{r}}{r^5}\right] \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left[\vec{d} - 3(\vec{d}\cdot\hat{r})\,\hat{r}\right]\end{aligned}$$

(b) $$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\hat{r}\,(\vec{d}\cdot\hat{r}) - \vec{d}\right]$$

$$\begin{aligned}\vec{F}_{charge} &= q\,\vec{E} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{d}\cdot\hat{r})\,\hat{r} - \vec{d}\right]\end{aligned}$$

(c) $$\vec{F}_{charge} = -\vec{F}_{dipole}.$$

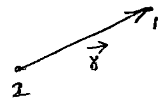
Yes, it confirms Newton's third law.

HW-07, prob 2

(a)

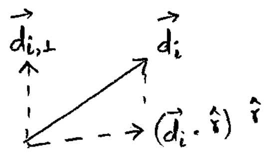
$$\vec{E}_1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\,\hat{r}\,(\vec{d}_1\cdot\hat{r}) - \vec{d}_1\right]$$

$$\begin{aligned} U_{12} &= -\vec{d}_2\cdot\vec{E} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\,(\vec{d}_2\cdot\hat{r})(\vec{d}_1\cdot\hat{r}) - \vec{d}_2\cdot\vec{d}_1\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\vec{d}_1\cdot\vec{d}_2 - 3\,(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\right] \end{aligned}$$



(b)

$$\begin{aligned} \vec{F}_{12} &= -\vec{\nabla}U_{12} \\ &= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left\{\frac{1}{r^3}\left[\vec{d}_1\cdot\vec{d}_2 - 3(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\right]\right\} \\ &= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{d}_1\cdot\vec{d}_2}{r^3}\right) + \frac{3}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{(\vec{d}_1\cdot\vec{r})(\vec{d}_2\cdot\vec{r})}{r^5}\right) \\ &= \frac{3}{4\pi\epsilon_0}\frac{(\vec{d}_1\cdot\vec{d}_2)}{r^5}\vec{r} + \frac{3}{4\pi\epsilon_0}\frac{\vec{d}_1(\vec{d}_2\cdot\vec{r})}{r^5} + \frac{3}{4\pi\epsilon_0}\frac{(\vec{d}_1\cdot\vec{r})\,\vec{d}_2}{r^5} \\ &\quad - \frac{15}{4\pi\epsilon_0}\frac{(\vec{d}_1\cdot\vec{r})(\vec{d}_2\cdot\vec{r})}{r^7}\vec{r} \\ &= \frac{1}{4\pi\epsilon_0}\frac{3}{r^4}\left[(\vec{d}_1\cdot\vec{d}_2)\,\hat{r} + (\vec{d}_2\cdot\hat{r})\,\vec{d}_1 + (\vec{d}_1\cdot\hat{r})\,\vec{d}_2 - 5(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\,\hat{r}\right] \end{aligned}$$



(c) writing

$$\vec{d}_1 = (\vec{d}_1\cdot\hat{r})\,\hat{r} + \vec{d}_{1,\perp}$$

$$\vec{d}_2 = (\vec{d}_2\cdot\hat{r})\,\hat{r} + \vec{d}_{2,\perp}$$

we have

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{3}{r^4}\left[(\vec{d}_1\cdot\vec{d}_2)\,\hat{r} - 3(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\,\hat{r} + \underbrace{(\vec{d}_2\cdot\hat{r})\,\vec{d}_{1,\perp} + (\vec{d}_1\cdot\hat{r})\,\vec{d}_{2,\perp}}_{\text{non-central.}}\right]$$

The force is not central.

(d) Using (b)

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{3}{r^4}\left[(\vec{d}_1\cdot\vec{d}_2)\,\hat{r} + (\vec{d}_2\cdot\hat{r})\,\vec{d}_1 + (\vec{d}_1\cdot\hat{r})\,\vec{d}_2 - 5(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\,\hat{r}\right]$$

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{3}{r^4}\left[-(\vec{d}_1\cdot\vec{d}_2)\,\hat{r} - (\vec{d}_2\cdot\hat{r})\,\vec{d}_1 - (\vec{d}_1\cdot\hat{r})\,\vec{d}_2 + 5(\vec{d}_1\cdot\hat{r})(\vec{d}_2\cdot\hat{r})\,\hat{r}\right]$$

by setting $\hat{r} \to -\hat{r}$

$$= -\vec{F}_{12}$$

Yes, they satisfy Newton's third law.

HW-07, prob 3

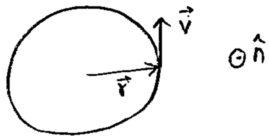
$$\rho_{eff} = -\vec{\nabla}\cdot\vec{P} \quad\Rightarrow\quad \frac{\partial\rho_{eff}}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{P}$$

$$\vec{j}_{eff} = \frac{\partial}{\partial t}\vec{P} + \vec{\nabla}\times\vec{M} \quad\Rightarrow\quad \vec{\nabla}\cdot\vec{j}_{eff} = \vec{\nabla}\cdot\frac{\partial\vec{P}}{\partial t} + \underbrace{\vec{\nabla}\cdot\vec{\nabla}\times\vec{M}}_{=0}$$

$$= \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{P}$$

Together $\dfrac{\partial\rho_{eff}}{\partial t} + \vec{\nabla}\cdot\vec{j}_{eff} = 0.$

HW-07, prob 4



$$\vec{\mu} = \hat{n}\,\frac{1}{2}\,q\,r\,v$$

$$= \hat{n}\,\frac{1}{\cancel{2}}\,q\,r\,\frac{\cancel{2}\pi r}{t}$$

$$= \hat{n}\,\frac{q}{t}\,\pi r^2$$

$$= \hat{n}\,I\,A$$

HW-07, prob 5

(a)
$$
\begin{aligned}
\rho_{eff} &= -\vec{\nabla}\cdot\vec{P} \\
&= -\vec{\nabla}\cdot\left[\alpha\,\vec{r}\,\theta(R-r)\right] \\
&= -\alpha\,(\vec{\nabla}\cdot\vec{r})\,\theta(R-r) - \alpha\,\vec{r}\cdot\vec{\nabla}\,\theta(R-r) \\
&= -3\alpha\,\theta(R-r) - \alpha\vec{r}\cdot(\vec{\nabla} r)\,\frac{\partial}{\partial r}\theta(R-r) \\
&= -3\alpha\,\theta(R-r) + \alpha\, r\,\delta(R-r) \\
&=
\end{aligned}
$$

$\vec{\nabla} r = \hat{r}$.

(b) $r < R$:

$$
\begin{aligned}
Q_{en} &= \int_{V(r)} d^3r'\,\rho_{eff}(\vec{r}\,') \\
&= 4\pi \int_0^{r<R} r'^2\,dr' \left[-3\alpha\,\theta(R-r') + \alpha\, r'\,\delta(R-r')\right] \\
&= -3\alpha\, 4\pi \int_0^{r<R} r'^2\,dr'\,\underbrace{\theta(R-r')}_{=1} + 4\pi\alpha \int_0^{r<R} r'^3\,dr'\,\delta(R-r')
\end{aligned}
$$

does not contribute because $r' \neq R$.

$$
\begin{aligned}
&= -3\alpha\, 4\pi\, \frac{r^3}{3} \\
&= -4\pi\alpha\, r^3 .
\end{aligned}
$$

$R < r$:

$$
\begin{aligned}
Q_{en} &= \int_{V(r)} d^3r'\,\rho_{eff}(\vec{r}\,') \\
&= 4\pi \int_0^{R<r} r'^2\,dr' \left[-3\alpha\,\theta(R-r') + \alpha\, r'\,\delta(R-r')\right] \\
&= -4\pi\, 3\alpha \int_0^{R} r'^2\,dr'\,\underbrace{\theta(R-r')}_{=1} + 4\pi\alpha \int_0^{R} r'^3\,dr'\,\delta(R-r') \\
&= -4\pi\, 3\alpha\, \frac{R^3}{3} + 4\pi\alpha R^3 . \\
&= 0 .
\end{aligned}
$$

(c)

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{en}.$$

$$E \, 4\pi r^2 = \frac{1}{\epsilon_0} Q_{en}.$$

$$\vec{E} = \hat{r} \frac{1}{4\pi\epsilon_0} \frac{Q_{en}}{r^2}$$

$$= \hat{r} \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \begin{cases} -4\pi\alpha r^3 & \text{if } r < R \\ 0 & \text{if } R < r. \end{cases}$$

$$= \begin{cases} -\frac{\alpha}{\epsilon_0} \vec{r}, & \text{if } r < R, \\ 0, & \text{if } R < r. \end{cases}$$