Homework No. 07 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Monday, 2014 Oct 27, 4:00 PM

1. (**30 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.9.)

   (a) The electric field of a point charge q at distance $\mathbf{r}$ is

   $$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}. \tag{1}$$

   The force on a point dipole in the presence of an electric field is

   $$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{2}$$

   Use these to find the force on a point dipole due to a point charge.

   (b) The electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by Eq. (4). The force on a point charge in the presence of an electric field is

   $$\mathbf{F} = q\mathbf{E}. \tag{3}$$

   Use these to find the force on a point charge due to a point dipole.

   (c) Confirm that above two forces are equal in magnitude and opposite in direction, as per Newton's third law.

2. (**40 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.8.)
   We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression

   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{4}$$

   The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

   $$U = -\mathbf{d}\cdot\mathbf{E}. \tag{5}$$

   Further, the force between the two dipoles is given by

   $$\mathbf{F} = -\boldsymbol{\nabla}U. \tag{6}$$

   Use these expressions to derive

(a) the interaction energy between two point dipoles separated by distance $\mathbf{r}$ to be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[\mathbf{d}_1\cdot\mathbf{d}_2 - 3\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\big]. \tag{7}$$

(b) the force between the two dipoles to be

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}\big[(\mathbf{d}_1\cdot\mathbf{d}_2)\,\hat{\mathbf{r}} + (\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + (\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 - 5\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\big]. \tag{8}$$

(c) Are the forces central? That is, is the force in the direction of $\mathbf{r}$?

(d) Are the forces on the dipole equal in magnitude and opposite in direction? That is, do they satisfy Newton's third law?

3. (**10 points.**) Show that the effective charge density, ρ_{eff}, and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{9}$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla}\times\mathbf{M}, \tag{10}$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla}\cdot\mathbf{j}_{\text{eff}} = 0. \tag{11}$$

4. (**10 points.**) The magnetic dipole moment of charge q_a moving with velocity $\mathbf{v}_a$ is

$$\boldsymbol{\mu} = \frac{1}{2}q_a\mathbf{r}_a\times\mathbf{v}_a, \tag{12}$$

where $\mathbf{r}_a$ is the position of the charge. For a charge moving along a circular orbit of radius r_a, with constant speed v_a, deduce the magnetic moment

$$\boldsymbol{\mu} = IA\,\hat{\mathbf{n}}, \qquad I = \frac{q_a}{\Delta t}\frac{v_a\Delta t}{2\pi r_a} \qquad A = \pi r_a^2, \tag{13}$$

where $\hat{\mathbf{n}}$ points along $\mathbf{r}_a\times\mathbf{v}_a$.

5. (**30 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.10.)
Consider a uniformly polarized sphere described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R-r). \tag{14}$$

(a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R-r) + \alpha r\delta(r-R). \tag{15}$$

(b) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r\, \rho_{\text{eff}}(\mathbf{r}) \tag{16}$$

for $r < R$ and $r > R$.

(c) Use Gauss's law to find the electric field to be

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{\varepsilon_0}\,\mathbf{r}, & \text{if} \quad r < R, \\ 0, & \text{if} \quad R < r. \end{cases} \tag{17}$$