Homework No. 08 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Nov 5, 4:00 PM

1. **(10 points.)** Verify the identity
$$\phi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\psi)-\psi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\phi)=\boldsymbol{\nabla}\cdot[\lambda(\phi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\phi)], \tag{1}$$
which is a slight generalization of what is known as Green's second identity.

2. **(10 points.)** Show that
$$\bar{\delta}(x)=-x\frac{d}{dx}\delta(x) \tag{2}$$
is also a model for the δ-function by showing that
$$\int_{-\infty}^{\infty}dx\,\bar{\delta}(x)f(x)=f(0). \tag{3}$$
Hint: Integrate by parts.

3. **(40 points.)** Verify that
$$\frac{d}{dz}|z|=\theta(z)-\theta(-z), \tag{4}$$
where $\theta(z)=1$, if $z>0$, and 0, if $z<0$. Further, verify that
$$\frac{d^2}{dz^2}|z|=2\,\delta(z). \tag{5}$$
Also, argue that, for a well defined function $f(z)$, the replacement
$$f(z)\delta(z)=f(0)\delta(z) \tag{6}$$
is justified. Using Eq. (4), Eq. (5), and Eq. (6), verify (by substituting the solution into the differential equation) that
$$g(z)=\frac{1}{2k}\,e^{-k|z|} \tag{7}$$
is a particular solution of the differential equation
$$\left(-\frac{d^2}{dz^2}+k^2\right)g(z)=\delta(z). \tag{8}$$

HW-08, prob 1

$$\phi\, \vec{\nabla}\cdot(\lambda \vec{\nabla}\psi) = \vec{\nabla}\cdot[\phi\lambda\vec{\nabla}\psi] - (\vec{\nabla}\phi)\cdot\lambda(\vec{\nabla}\psi)$$

$$\psi\, \vec{\nabla}\cdot(\lambda \vec{\nabla}\phi) = \vec{\nabla}\cdot[\psi\lambda\vec{\nabla}\phi] - (\vec{\nabla}\psi)\cdot\lambda(\vec{\nabla}\phi)$$

Subtracting we have.

$$\phi\, \vec{\nabla}\cdot(\lambda\vec{\nabla}\psi) - \psi\, \vec{\nabla}\cdot(\lambda\vec{\nabla}\phi) = \vec{\nabla}\cdot\left[\lambda\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right)\right].$$

HW-08, prob 2

$$\int_{-\infty}^{+\infty} dx\; \bar{\delta}(x)\, f(x) = -\int_{-\infty}^{+\infty} dx\; x\left[\frac{d}{dx}\delta(x)\right] f(x)$$

$$= \underbrace{-\int_{-\infty}^{+\infty} dx\, \frac{d}{dx}\left[x\, f(x)\, \delta(x)\right]}_{=0,\ \text{because } \delta(x)=0 \text{ at } \pm\infty.} + \int_{-\infty}^{+\infty} dx\; \delta(x)\, \frac{d}{dx}\left(x\, f(x)\right)$$

$$= \int_{-\infty}^{+\infty} dx\; \delta(x)\left[f(x) + x\frac{d}{dx}f(x)\right]$$

$$= f(0) + \underbrace{x\frac{d}{dx}f(x)\Big|_{x=0}}_{=0}$$

$$= f(0).$$

HW-08, prob 3

$$|z| = \begin{cases} z, & \text{if } z>0, \\ -z, & \text{if } z<0. \end{cases}$$

$$\frac{d}{dz}|z| = \begin{cases} 1, & \text{if } z>0, \\ -1, & \text{if } z<0, \end{cases}$$

$$= \theta(z) - \theta(-z)$$

$$\frac{d^2}{dz^2}|z| = \frac{d}{dz}\theta(z) - \frac{d}{dz}\theta(-z)$$

$$= \delta(z) + \delta(z) = 2\delta(z)$$

$$LHS = \left(-\frac{d^2}{dz^2} + k^2\right) g(z)$$

$$= \left(-\frac{d^2}{dz} + k^2\right) \frac{1}{2k} e^{-k|z|}$$

$$= -\frac{d}{dz}\left[\frac{(-k)}{2k} e^{-k|z|} \frac{d}{dz}|z|\right] + k^2 g$$

$$= -k^2 g \underbrace{\left(\frac{d}{dz}|z|\right)^2}_{=1} + \frac{1}{2} e^{-k|z|} \frac{d^2}{dz^2}|z| + k^2 g$$

$$= -k^2 g + \frac{1}{2} e^{-k|z|} 2\delta(z) + k^2 g$$

$$= e^{-k|z|}\delta(z)$$

$$= \delta(z).$$

HW-08, prob 4

(a) $$G_R(t-t') = -\frac{1}{\omega}\, \theta(t-t')\, \sin\omega(t-t')$$

$$\frac{d}{dt} G_R(t-t') = -\frac{1}{\omega}\, \delta(t-t')\sin\omega(t-t') - \theta(t-t')\cos\omega(t-t')$$

$$\frac{d^2}{dt^2} G_R(t-t') = -\frac{1}{\omega}\left[\frac{d}{dt}\delta(t-t')\right]\sin\omega(t-t') - \delta(t-t')\cos\omega(t-t') + \omega\,\theta(t-t')\sin\omega(t-t')$$

$$- \delta(t-t')\cos\omega(t-t')$$

$$= -\frac{\sin\omega(t-t')}{\omega(t-t')}\,(t-t')\frac{d}{dt}\delta(t-t') - 2\,\delta(t-t')\cos\omega(t-t')$$

$$-\omega^2 G_R(t-t')$$

$$-\left(\frac{d^2}{dt^2}+\omega^2\right) G_R(t-t') = +(t-t')\frac{d}{dt}\delta(t-t') + 2\,\delta(t-t')$$ (using prob. 2)

$$= -\delta(t-t') + 2\delta(t-t')$$

$$= \delta(t-t') \quad ✓$$

(b) $$G_A(t-t') = \frac{1}{\omega}\,\theta(t'-t)\,\sin\omega(t-t')$$

$$\frac{d}{dt} G_A(t-t') = -\frac{1}{\omega}\delta(t-t')\sin\omega(t-t') + \theta(t'-t)\cos\omega(t-t')$$

$$\frac{d^2}{dt^2} G_A(t-t') = -\frac{1}{\omega}\frac{d}{dt}\delta(t-t')\sin\omega(t-t') - \delta(t-t')\cos\omega(t-t')$$

$$- \delta(t-t')\cos\omega(t-t') - \omega\,\theta(t'-t)\sin\omega(t-t')$$

Thus.

$$-\left(\frac{d^2}{dt^2}+\omega^2\right) G_A(t-t') = \frac{\sin\omega(t-t')}{\omega(t-t')}\,(t-t')\,\frac{d}{dt}\delta(t-t') + 2\,\delta(t-t')\cos\omega(t-t')$$

$$= (t-t')\,\frac{d}{dt}\delta(t-t') + 2\,\delta(t-t')$$

$$= -\,\delta(t-t') + 2\,\delta(t-t')$$

$$= \delta(t-t') \quad \checkmark$$

(c)

$$G_R(t-t') - G_A(t-t') = -\frac{1}{\omega}\,\theta(t-t')\sin\omega(t-t') - \frac{1}{\omega}\,\theta(t'-t)\sin\omega(t-t')$$

$$= -\frac{1}{\omega}\left[\theta(t-t') + \theta(t'-t)\right]\sin\omega(t-t')$$

$$= -\frac{1}{\omega}\sin\omega(t-t')$$

$$\frac{d^2}{dt^2}\left[G_R(t-t') - G_A(t-t')\right] = +\,\omega\sin\omega(t-t')$$

$$= -\,\omega^2\left[G_R - G_A\right]$$

$$\Rightarrow\ -\left(\frac{d^2}{dt^2}+\omega^2\right)\left[G_R - G_A\right] = 0$$