# Homework No. 08 (2014 Fall)

## PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Nov 5, 4:00 PM

1. (**10 points.**) Verify the identity

$$\phi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\psi)-\psi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\phi)=\boldsymbol{\nabla}\cdot[\lambda(\phi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\phi)], \tag{1}$$

which is a slight generalization of what is known as Green's second identity. Here $\phi$, $\psi$, and $\lambda$, are position dependent functions.

2. (**10 points.**) Show that

$$\bar{\delta}(x)=-x\frac{d}{dx}\delta(x) \tag{2}$$

is also a model for the $\delta$-function by showing that

$$\int_{-\infty}^{\infty}dx\,\bar{\delta}(x)f(x)=f(0). \tag{3}$$

Hint: Integrate by parts.

3. (**40 points.**) Verify that

$$\frac{d}{dz}|z|=\theta(z)-\theta(-z), \tag{4}$$

where $\theta(z)=1$, if $z>0$, and 0, if $z<0$. Further, verify that

$$\frac{d^2}{dz^2}|z|=2\,\delta(z). \tag{5}$$

Also, argue that, for a well defined function $f(z)$, the replacement

$$f(z)\delta(z)=f(0)\delta(z) \tag{6}$$

is justified. Using Eq. (4), Eq. (5), and Eq. (6), verify (by substituting the solution into the differential equation) that

$$g(z)=\frac{1}{2k}\,e^{-k|z|} \tag{7}$$

is a particular solution of the differential equation

$$\left(-\frac{d^2}{dz^2}+k^2\right)g(z)=\delta(z). \tag{8}$$

4. (**70 points.**) A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t)=F(t), \tag{9}$$

where $\omega$ is the angular frequency of the oscillator and $F(t)$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{10}$$

The continuity conditions satisfied by the Green function are

$$\frac{d}{dt}G(t,t')\bigg|_{t=t'-\delta}^{t=t'+\delta}=-1 \tag{11}$$

and

$$G(t,t')\bigg|_{t=t'-\delta}^{t=t'+\delta}=0. \tag{12}$$

(a) Verify that a particular solution,

$$G_R(t-t')=-\frac{1}{\omega}\theta(t-t')\sin\omega(t-t'), \tag{13}$$

which is referred to as the retarded Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 2 and $\lim_{x\to\infty}\sin x/x=0$.

(b) Verify that another particular solution,

$$G_A(t-t')=\frac{1}{\omega}\theta(t'-t)\sin\omega(t-t'), \tag{14}$$

which is referred to as the advanced Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 2 and $\lim_{x\to\infty}\sin x/x=0$.

(c) Show that the difference of the two particular solutions above,

$$G_R(t-t')-G_A(t-t'), \tag{15}$$

satisfies the homogeneous differential equations

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G_0(t,t')=0. \tag{16}$$