Solutions

# Homework No. 09 (2014 Fall)

## PHYS 320: Electricity and Magnetism I

Due date: Friday, 2014 Nov 21, 4:00 PM

1. **(10 points.)** Interaction energy of a dipole $\mathbf{d}$ with an electric field $\mathbf{E}$ is

$$U = -\mathbf{d}\cdot\mathbf{E} = -dE\cos\theta. \tag{1}$$

The torque on the dipole due to the electric field is

$$\boldsymbol{\tau} = \mathbf{d}\times\mathbf{E}. \tag{2}$$

Force is a manifestation of the systems tendency to minimize its energy, and in this spirit torque is defined as,

$$\tau = -\frac{\partial}{\partial\theta}U = -dE\sin\theta. \tag{3}$$

Show that there is no inconsistency, in sign, between the two definitions of torque.

2. **(50 points.)** Consider the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z} + \varepsilon(z)k_\perp^2\right] g_\varepsilon(z,z') = \delta(z-z'), \tag{4}$$

for the case

$$\varepsilon(z) = \begin{cases}\varepsilon_2, & z<0,\\ \varepsilon_1, & 0<z,\end{cases} \tag{5}$$

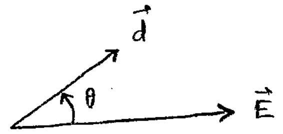
satisfying the boundary conditions

$$g_\varepsilon(-\infty,z') = 0, \tag{6a}$$
$$g_\varepsilon(+\infty,z') = 0. \tag{6b}$$

(a) Verify, by integrating Eq. (4) around $z=z'$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{7a}$$
$$\varepsilon(z)\frac{\partial}{\partial z}g_\varepsilon(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{7b}$$

HW-09, prob. 1




$$U = -\vec{d}\cdot\vec{E} = -dE\cos\theta$$

$$\vec{\tau} = \vec{d}\times\vec{E}$$
$$= -\vec{E}\times\vec{d}$$
(to point out that $\theta$ is measured with respect to $\vec{E}$)

Thus,
$$\tau = -Ed\sin\theta \quad \checkmark$$

HW-09, prob 2

$$\left[-\frac{d}{dz}\,\epsilon(z)\,\frac{d}{dz} + \epsilon(z)\,k_\perp^2\right] g_e(z,z') = \delta(z-z')$$

(a) Integrating about $z=z'$, we notice that to obtain the delta function on the right hand side we require (see class notes)

$$g_e(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0$$

and

$$\epsilon(z)\,\frac{\partial}{\partial z}\,g_e(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1.$$

(b) Integrating about $z=0$, we similarly have: (see class notes)

$$g_e(z,z')\Big|_{z=0-\delta}^{z=0+\delta} = 0$$

$$\epsilon(z)\,\frac{\partial}{\partial z}\,g_e(z,z')\Big|_{z=0-\delta}^{z=0+\delta} = 0.$$

(c) Let $z' < 0$.

$$g_\epsilon(z,z') = \begin{cases} A_1 e^{k_1 z} + \cancel{B_1}^{=0} e^{-k_1 z}, & z < z' < 0, \\ C_1 e^{k_1 z} + D_1 e^{-k_1 z}, & z' < z < 0, \\ \cancel{E_1}^{=0} e^{k_1 z} + F_1 e^{-k_1 z}, & z' < 0 < z. \end{cases}$$

$$g_\epsilon(+\infty, z') = 0 \quad \Rightarrow \quad E_1 = 0$$

$$g_\epsilon(-\infty, z') = 0 \quad \Rightarrow \quad B_1 = 0.$$

Using (9) in (8)a & (8)b.

$$F_1 - C_1 = D_1$$

$$\epsilon_1 F_1 + \epsilon_2 C_1 = \epsilon_2 D_1$$

$$\Rightarrow \quad F_1 = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} D_1, \qquad C_1 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} D_1$$

Using (9) in (7)(a) & (7)b.

$$C_1 e^{k_1 z'} + D_1 e^{-k_1 z'} - A_1 e^{k_1 z'} = 0$$

$$C_1 e^{k_1 z'} - D_1 e^{-k_1 z'} - A_1 e^{k_1 z'} = -\frac{1}{\epsilon_2 k_1}$$

$$\left\{ \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{k_1 z'} + e^{-k_1 z'} \right\} D_1 - A_1 e^{k_1 z'} = 0$$

$$\left\{ \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{k_1 z'} - e^{-k_1 z'} \right\} D_1 - A_1 e^{k_1 z'} = -\frac{1}{\epsilon_2 k_1}.$$

$$D_1 = \frac{1}{\epsilon_2} \frac{1}{2k_1} e^{k_1 z'}$$

$$A_1 = \frac{1}{\epsilon_2} \frac{1}{2k_1} \left\{ \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{k_1 z'} + e^{-k_1 z'} \right\}$$

Thus, we have.

$$A_1 = \frac{1}{\epsilon_2}\frac{1}{2k_1}\left\{ \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{k_1 z'} + e^{-k_1 z'} \right\}$$

$$B_1 = 0$$

$$C_1 = \frac{1}{\epsilon_2}\frac{1}{2k_1}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{k_1 z'}$$

$$D_1 = \frac{1}{\epsilon_2}\frac{1}{2k_1} e^{k_1 z'}$$

$$E_1 = 0$$

$$F_1 = \frac{1}{\epsilon_2}\frac{1}{2k_1}\frac{2\epsilon_2}{\epsilon_1+\epsilon_2} e^{k_1 z'}.$$

So, for $z'<0$, we have

$$g_e(z,z') = \begin{cases} \frac{1}{\epsilon_2}\frac{1}{2k_1} e^{-k_1(z'-z)} + \frac{1}{\epsilon_2}\frac{1}{2k_1}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{k_1 z} e^{+k_1 z'}, & z<z'<0, \\ \frac{1}{\epsilon_2}\frac{1}{2k_1} e^{-k_1(z-z')} + \frac{1}{\epsilon_2}\frac{1}{2k_1}\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1} e^{k_1 z} e^{k_1 z'}, & z'<z<0, \\ \frac{1}{\epsilon_2}\frac{1}{2k_1}\frac{2\epsilon_2}{\epsilon_1+\epsilon_2} e^{-k_1(z-z')}, & z'<0<z. \end{cases}$$

$$= \frac{1}{\epsilon_2}\frac{1}{2k_1} e^{-k_1|z-z'|} + \frac{1}{\epsilon_2}\frac{1}{2k_1}\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right) e^{-k_1|z|} e^{-k_1|z'|}, \quad z'<0.$$

(d) For $0<z'$, we write

$$g_e(z,z') = \begin{cases} A_2 e^{k_1 z} + \overset{=0}{\cancel{B_2}} e^{-k_1 z}, & z<0<z', \\ C_2 e^{k_1 z} + D_2 e^{-k_1 z}, & 0<z\leq z', \\ \overset{=0}{\cancel{E_2}} e^{k_1 z} + F_2 e^{-k_1 z}, & 0<z'<z. \end{cases}$$

Using ⑩ in ⑧ a and ⑧ b.

$$C_2 + D_2 - A_2 = 0$$

$$\epsilon_1 C_2 - \epsilon_1 D_2 - \epsilon_2 A_2 = 0$$

$$D_2 - A_2 = -C_2$$

$$\epsilon_1 D_2 + \epsilon_2 A_2 = \epsilon_1 C_2 .$$

$$\Rightarrow \quad D_2 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} C_2$$

$$A_2 = \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} C_2$$

Using ⑩ in ⑦(a) and ⑦(b)

$$F_2 e^{-k_1 z'} - C_2 e^{k_1 z'} - D_2 e^{-k_1 z'} = 0$$

$$-F_2 e^{-k_1 z'} - C_2 e^{k_1 z'} + D_2 e^{-k_1 z'} = -\frac{1}{\epsilon_1 k_1}$$

$$F_2 e^{-k_1 z'} - C_2 \left\{ e^{k_1 z'} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_1 z'} \right\} = 0$$

$$-F_2 e^{-k_1 z'} - C_2 \left\{ e^{k_1 z'} - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_1 z'} \right\} = -\frac{1}{\epsilon_1} \frac{1}{k_1}$$

$$F_2 = +\frac{1}{\epsilon_1} \frac{1}{2k_1} \left\{ e^{k_1 z'} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_1 z'} \right\}$$

$$C_2 = +\frac{1}{\epsilon_1} \frac{1}{2k_1} e^{-k_1 z'}$$

Thus,

$$A_2 = \frac{1}{\epsilon_1} \frac{1}{2k_1} \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} e^{-k_1 z'}$$

$$B_2 = 0$$

$$C_2 = \frac{1}{\epsilon_1} \frac{1}{2k_1} e^{-k_1 z'}$$

$$D_2 = \frac{1}{\epsilon_1} \frac{1}{2k_1} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_1 z'}$$

$$E_2 = 0$$

$$F_2 = \frac{1}{\epsilon_1} \frac{1}{2k_1} \left\{ e^{k_1 z'} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_1 z'} \right\}$$

So,

$$g_\epsilon(z,z') = \begin{cases} \frac{1}{\epsilon_1}\frac{1}{2k_1}\frac{2\epsilon_1}{\epsilon_1+\epsilon_2} e^{-k_1(z'-z)}, & z<0<z', \\ \frac{1}{\epsilon_1}\frac{1}{2k_1} e^{-k_1(z'-z)} + \frac{1}{\epsilon_1}\frac{1}{2k_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} e^{-k_1 z} e^{-k_1 z'}, & 0<z<z', \\ \frac{1}{\epsilon_1}\frac{1}{2k_1} e^{-k_1(z-z')} + \frac{1}{\epsilon_1}\frac{1}{2k_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} e^{-k_1 z} e^{-k_1 z'}, & 0<z'<z. \end{cases}$$

$$= \frac{1}{\epsilon_1}\frac{1}{2k_1} e^{-k_1|z-z'|} + \frac{1}{\epsilon_1}\frac{1}{2k_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2} e^{-k_1|z|} e^{-k_1|z'|}, \quad 0<z'.$$

(e) Already verified in (c) and (d).

prob 3, HW-09

$\vec{r}_0 = d\,\hat{z}$ $\quad q_i = -q$.

$\vec{r}_i = -d\,\hat{z} = -\vec{r}_0$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} + \frac{q_i}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} - \frac{q}{4\pi\epsilon_0}\frac{\vec{r}+\vec{r}_0}{|\vec{r}+\vec{r}_0|^3}.$$

$\vec{r} = d\,\hat{x} + 2d\,\hat{z}$

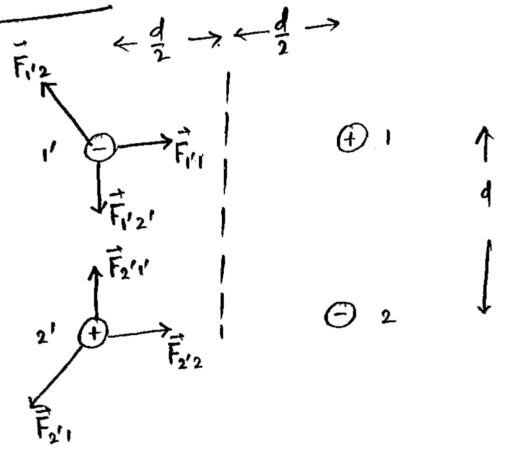
$\vec{r}-\vec{r}_0 = d\,\hat{x} + d\,\hat{z} \Rightarrow |\vec{r}-\vec{r}_0| = \sqrt{2}\,d$

$\vec{r}+\vec{r}_0 = d\,\hat{x} + 3d\,\hat{z} \Rightarrow |\vec{r}+\vec{r}_0| = \sqrt{10}\,d.$

$$\vec{E}(d\,\hat{x}+2d\,\hat{z}) = \frac{q}{4\pi\epsilon_0}\left[\frac{d\hat{x}+d\hat{z}}{(\sqrt{2}\,d)^3} - \frac{d\hat{x}+3d\hat{z}}{(\sqrt{10}\,d)^3}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\left[\left\{\frac{1}{(\sqrt{2})^3} - \frac{1}{(\sqrt{10})^3}\right\}\hat{i} + \left\{\frac{1}{(\sqrt{2})^3} - \frac{3}{(\sqrt{10})^3}\right\}\hat{z}\right]$$

prob 4, HW-09




$$\vec{F}_{tot} = \text{Total force on } 1' \text{ and } 2'$$

$$= \vec{F}_{1'1} + \vec{F}_{2'2} + \vec{F}_{1'2} + \vec{F}_{2'1}$$

$$= \hat{i}\,\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{d^2} + \frac{1}{d^2} - \frac{\cos 45}{d^2} - \frac{\cos 45}{d^2}\right]$$

$$= \hat{i}\,\frac{q^2}{4\pi\epsilon_0 d^2}\left[2 - \frac{2}{\sqrt{2}}\right]$$

$$= \hat{i}\,\frac{q^2}{4\pi\epsilon_0 d^2}\,2\left(1 - \frac{1}{\sqrt{2}}\right)$$