Homework No. 10 (2014 Fall)

PHYS 320: Electricity and Magnetism I

Due date: Wednesday, 2014 Dec 3, 4:00 PM

1. **(10 points.)** Using the series representation for Bessel functions,

$$J_m(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left(\frac{t}{2}\right)^{m+2n}, \tag{1}$$

prove the relation

$$J_m(t) = (-1)^m J_{-m}(t). \tag{2}$$

Hint: Break the sum on n into two parts. Note that the gamma function $\Gamma(z)$, which generalizes the factorial,

$$n! = \Gamma(n+1), \qquad \Gamma(z+1) = z\Gamma(z), \tag{3}$$

beyond positive integers, satisfies

$$\frac{1}{\Gamma(-k)} = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{4}$$

2. **(20 points.)** Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{5}$$

to prove the recurrence relations

$$2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{6a}$$

$$2\frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{6b}$$

3. **(10 points.)** Using the recurrence relations of Eq. (6), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \tag{7}$$

and from this derive the differential equation satisfied by $J_m(t)$.

HW-10, prob 1

$$J_m(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{t}{2}\right)^{m+2n}$$

$$(-1)^m J_{-m}(t) = (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(-m+n)!} \left(\frac{t}{2}\right)^{-m+2n}$$

$$= \underbrace{(-1)^m \sum_{n=0}^{m-1} \frac{(-1)^n}{n!\,(-m+n)!} \left(\frac{t}{2}\right)^{-m+2n}}_{=0 \ \left(\because \frac{1}{(-m+n)!} = \frac{1}{\Gamma(-m+n+1)} = 0\right)} + (-1)^m \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,(-m+n)!} \left(\frac{t}{2}\right)^{-m+2n}$$

$$= (-1)^m \sum_{n=m}^{\infty} \frac{(-1)^n}{n!\,(-m+n)!} \left(\frac{t}{2}\right)^{-m+2n}.$$

let $-m+n = n'$

$$= (-1)^m \sum_{n'=0}^{\infty} \frac{(-1)^{m+n'}}{(m+n')!\,n'!} \left(\frac{t}{2}\right)^{-m+2(n'+m)}$$

$$= \sum_{n'=0}^{\infty} \frac{(-1)^{n'}}{n'!\,(n'+m)!} \left(\frac{t}{2}\right)^{m+2n'}$$

$$= J_m(t)$$

HW-10, prob 2

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha - im\alpha}$$

$$2\frac{d}{dt} J_m(t) = 2\frac{d}{dt}\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha - im\alpha}$$

$$= 2\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, i\cos\alpha\, e^{it\cos\alpha - im\alpha}$$

$$= 2\frac{1}{i^{m-1}}\int_0^{2\pi}\frac{d\alpha}{2\pi}\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right) e^{it\cos\alpha}\, e^{-im\alpha}$$

$$= \frac{1}{i^{m-1}}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha - i(m-1)\alpha} + \frac{i^2}{i^{m+1}}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}\, e^{-i(m+1)\alpha}$$

$$= J_{m-1}(t) - J_{m+1}(t)$$

$$J_{m-1}(t) + J_{m+1}(t) = \frac{1}{i^{m-1}}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}\, e^{-i(m-1)\alpha} + \frac{1}{i^{m+1}}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}\, e^{-i(m+1)\alpha}$$

$$= (-2)\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}\, e^{-im\alpha}\sin\alpha.$$

$$= -2\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\left(\frac{1}{-it}\frac{d}{d\alpha}e^{it\cos\alpha}\right) e^{-im\alpha}.$$

Uses $\frac{d}{d\alpha}e^{it\cos\alpha} = -i\sin\alpha\, e^{it\cos\alpha}$.

$$= \frac{2}{it}\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,\underbrace{\frac{d}{d\alpha}\left(e^{it\cos\alpha}\, e^{-im\alpha}\right)}_{=0} - \frac{2}{it}\frac{1}{i^m}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}\frac{d}{d\alpha}e^{-im\alpha}$$

$$= -\frac{2}{it}(-im)\, J_m(t)$$

$$= \frac{2m}{t} J_m(t)$$

HW-10, prob 3

We have
$$J_{m-1}(t) - J_{m+1}(t) = 2\frac{d}{dt}J_m(t)$$
$$J_{m-1}(t) + J_{m+1}(t) = \frac{2m}{t}J_m(t)$$

Adding and subtracting we have.
$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t) \quad \text{— (i)}$$
$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t) \quad \text{— (ii)}$$

Operating the respective operator again
$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(-\frac{d}{dt} + \frac{m-1}{t}\right) J_{m-1}(t) = J_m(t) \quad \text{— (iii)}$$
and
$$\left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right) J_{m+1}(t) = J_m(t) \quad \text{— (iv)}$$

Using (iii) we have.
$$-\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) + J_m(t) = 0$$
$$\left[\frac{d^2}{dt^2} + \frac{d}{dt}\frac{m}{t} - \frac{m-1}{t}\frac{d}{dt} - \frac{(m-1)m}{t^2} + 1\right] J_m(t) = 0$$
$$\left[\frac{d^2}{dt^2} - \frac{m}{t^2} + \frac{m}{t}\frac{d}{dt} - \frac{m-1}{t}\frac{d}{dt} - \frac{m^2-m}{t^2} + 1\right] J_m(t) = 0$$
$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{m^2}{t^2} + 1\right] J_m(t) = 0$$