Date: 2014 Aug 18

(1) Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0} \qquad Q = \int d\vec{r}\, \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint d\vec{l} \cdot \vec{E} = -\frac{\partial}{\partial t} \Phi_B$$

$$\Phi_B = \int_{surface} \vec{B} \cdot d\vec{a}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \Phi_E + \mu_0 I$$

$$\Phi_E = \int_{surface} \vec{B} \cdot d\vec{a}$$

$$I = \int_{surface} \vec{J} \cdot d\vec{a}$$

$$\vec{D} = \epsilon_0 \vec{E}$$

$$\vec{B} = \mu_0 \vec{H}$$

(2) Photons are a particular manifestation of electric and magnetic fields.

1 Vectors

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

~~Z~~ Let $\hat{e}_1 = \hat{x}$
$\hat{e}_2 = \hat{y}$
$\hat{e}_3 = \hat{z}$

$A_i \equiv (A_1, A_2, A_3)$

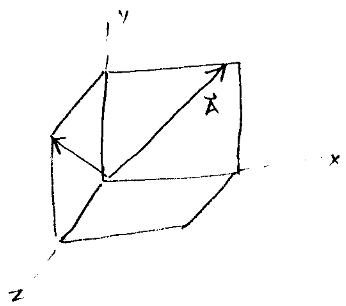
→ an array of numbers.

→ components of a vector, tensor of rank 1.

2 $\vec{C} = \vec{A} \pm \vec{B}$

$\vec{A} \cdot \vec{B}$

$\vec{A} \times \vec{B}$

3 

$\vec{A} = (1, 1, 0) \qquad |\vec{A}| = \sqrt{2}$

$\vec{B} = (0, 1, 1) \qquad |\vec{B}| = \sqrt{2}$

$\vec{A} \cdot \vec{B} = 1 = AB \cos\theta$

$1 = 2 \cos\theta$

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ.$

4 Linear transformat of a vector

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' = \cos\theta \; x + \sin\theta \; y$$

$$y' = -\sin\theta \; x + \cos\theta \; y.$$

$$x'^2 + y'^2 = (\cos\theta\, x + \sin\theta\, y)^2 + (-\sin\theta\, x + \cos\theta\, y)^2$$

$$= x^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + y^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{=1} - 2xy \sin\theta\cos\theta + 2xy \sin\theta\cos\theta$$

$$= x^2 + y^2.$$

5 $$x'_i = R_{ij}\, x_j = \sum_{j=1}^{2} R_{ij}\, x_j$$ ← summation convention, differentiate free index with dummy index

$\underline{\underline{i=1}}$ $$x'_1 = R_{1j}\, x_j = R_{11}\, x_1 + R_{12}\, x_2.$$

$\underline{\underline{i=2}}$ $$x'_2 = R_{2j}\, x_j = R_{21}\, x_1 + R_{22}\, x_2.$$

$$R_{ij} \equiv \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

6
$$\vec{x}' \cdot \vec{x}' = x_1' x_1' + x_2' x_2'$$
$$= x_i' x_i'$$
$$= (R_{ij} x_j)(R_{ik} x_k)$$
$$= R_{ij} R_{ik} x_j x_k.$$

7
$$\vec{x} \cdot \vec{x} = x_1 x_1 + x_2 x_2.$$
$$= x_j x_j$$
$$= x_j \delta_{jk} x_k.$$

8 Kronecker delta
$$\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

9
$$\vec{x}' \cdot \vec{x}' = \vec{x} \cdot \vec{x} \qquad \text{(definition of a vector)}$$
$$\Rightarrow \quad R_{ij} R_{ik} = \delta_{jk}.$$

(10) $$\overleftrightarrow{R} = R_{ij} \equiv \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$\overleftrightarrow{R}^{T} = R_{ji} = \begin{pmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{pmatrix}$$

(11) $$\overleftrightarrow{C} = \overleftrightarrow{M} \cdot \overleftrightarrow{N} = M_{ji} N_{ik}$$

$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} N_{11} + M_{12} N_{21} & M_{11} N_{12} + M_{12} N_{22} \\ M_{21} N_{11} + M_{22} N_{21} & \end{pmatrix}$$

$$= \begin{pmatrix} M_{1i} N_{i1} & M_{1i} N_{i2} \\ M_{2i} N_{i1} & M_{2i} N_{i2} \end{pmatrix}$$

$$\overleftrightarrow{C} = \overleftrightarrow{M} \cdot \overleftrightarrow{N}$$

$$C_{jk} = M_{ji} N_{ik}$$

(12)
$$R_{ij} R_{ik} = \delta_{jk}$$
$$\left(R^T\right)_{ji} R_{ik} = \delta_{jk}$$
$$\overleftrightarrow{R}^T \cdot \overleftrightarrow{R} = \overleftrightarrow{1}$$

(13) Rotation in arbitrary dimension.
$$x'_i = R_{ij} x_j$$
$$R_{ij} R_{ik} = \delta_{ik}$$

(14) Vector
$$A'_i = R_{ij} A_j$$
tensor
$$B'_{ij} = R_{im} R_{jn} B_{mn}$$

(15)
$$\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i=1,\ j=2,\ k=3 \text{ and even permutations} \\ -1 & \text{if } i=1,\ j=3,\ k=2 \text{ and even "} \\ 0 & \text{otherwise.} \end{cases}$$

16 $\vec{A} \cdot \vec{B} = A_i B_i$

$= A_i \delta_{ij} B_j$

17 $\vec{C} = \vec{A} \times \vec{B}$

$= (A_2 B_3 - A_3 B_2)\hat{x} + (A_3 B_1 - A_1 B_3)\hat{y} + (A_1 B_2 - A_2 B_1)\hat{z}$

$C_1 = A_2 B_3 - A_3 B_2$

$C_2 = A_3 B_1 - A_1 B_3$

$C_3 = A_1 B_2 - A_2 B_1$

$\epsilon_{ijk} A_j B_k = ?$

$i=1 \qquad \epsilon_{1jk} A_j B_k = \cancel{\epsilon_{11k} A_1 B_k} + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$

$= \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$

$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2$

$= A_2 B_3 - A_3 B_2$