

Date: 20th Aug 20

①

- ① $A_i \rightarrow$ components of a vector $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + \dots$
 $B_{ij} \rightarrow$ components of a matrix. $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \vec{B}$

- ② $\vec{B} \equiv B_{ij} \quad \vec{B}^T \equiv B_{ji}$

③ $\vec{C} = \vec{M} \cdot \vec{N}$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} N_{11} + M_{12} N_{21} & M_{11} N_{12} + M_{12} N_{22} \\ M_{21} N_{11} + M_{22} N_{21} & M_{21} N_{12} + M_{22} N_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^2 M_{1k} N_{k1} & \sum_{k=1}^2 M_{1k} N_{k2} \\ \sum_{k=1}^2 M_{2k} N_{k1} & \sum_{k=1}^2 M_{2k} N_{k2} \end{pmatrix}$$

$$= \begin{pmatrix} M_{1k} N_{k1} & M_{1k} N_{k2} \\ M_{2k} N_{k1} & M_{2k} N_{k2} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= M_{1k} N_{k1} \\ C_{12} &= M_{1k} N_{k2} \\ C_{21} &= M_{2k} N_{k1} \\ C_{22} &= M_{2k} N_{k2} \end{aligned}$$

$$C_{ij} = M_{ik} N_{kj}$$

i and $j \rightarrow$ free index
 $k \rightarrow$ dummy index.

④ Note that :

$$(i) \delta_{ik} \delta_{kj} = \delta_{ij}$$

$$(vi) \delta_{im} b_m = b_i$$

$$(ii) \delta_{ij} = \delta_{ji}$$

$$(iii) \delta_{ii} = 3$$

$$(iv) \epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$$

$$(v) \epsilon_{iik} = 0$$

$$\textcircled{5} \quad \vec{A} \cdot \vec{B} = A_i B_i = A_i \delta_{ij} B_j = (A_1 \ A_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\vec{C} = \vec{A} \times \vec{B} \Rightarrow C_i = \epsilon_{ijk} A_j B_k$$

$$\begin{aligned} \textcircled{6} \quad [\vec{A} \times (\vec{B} \times \vec{C})]_i &= \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k \\ &= \epsilon_{ijk} A_j \epsilon_{kmn} B_m C_n \\ &= \epsilon_{ijk} \epsilon_{kmn} A_j B_m C_n \end{aligned}$$

⑦ Identity

$$\begin{aligned} \epsilon_{ijk} \epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il} (\delta_{jm} \delta_{kn} - \delta_{km} \delta_{jn}) \\ &\quad - \delta_{im} (\delta_{jl} \delta_{kn} - \delta_{kl} \delta_{jn}) \\ &\quad + \delta_{in} (\delta_{jl} \delta_{km} - \delta_{kl} \delta_{jm}) \end{aligned}$$

$$\begin{aligned} \epsilon_{ijk} \epsilon_{kmn} &= \delta_{ik} (\delta_{jm} \delta_{kn} - \delta_{km} \delta_{jn}) \\ &\quad - \delta_{im} (\delta_{jk} \delta_{kn} - \delta_{kk} \delta_{jn}) \\ &\quad + \delta_{in} (\delta_{jk} \delta_{km} - \delta_{kk} \delta_{jm}) \\ &= \delta_{jm} \delta_{in} - \delta_{im} \delta_{jn} \\ &\quad - \delta_{im} \delta_{jn} + 3 \delta_{im} \delta_{jn} \\ &\quad + \delta_{in} \delta_{jm} - 3 \delta_{in} \delta_{jm} \\ &= \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \end{aligned}$$

$$\begin{aligned} \textcircled{8} \quad \vec{A} \times (\vec{B} \times \vec{C}) &= \epsilon_{ijk} A_j C_{km} B_m C_n \\ &= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) A_j B_m C_n \\ &= A_j B_i C_j - A_j B_j C_i \\ &= \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \end{aligned}$$

$$\begin{aligned}
 \textcircled{9} \quad (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= \epsilon_{ijk} A_j B_k \epsilon_{imn} C_m D_n \\
 &= (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) A_j B_k C_m D_n \\
 &= A_j B_k C_j D_k - A_j B_k C_k D_j \\
 &= (\vec{A} \cdot \vec{C}) (\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D}) (\vec{B} \cdot \vec{C})
 \end{aligned}$$