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① Variation

$$f(x+\delta x) = f(x) + \delta x \frac{df}{dx} + \frac{1}{2}\delta x^2 \frac{d^2 f}{dx^2} + \cdots$$

$$= f(x) + \delta x \frac{df}{dx} \quad \text{(upto leading order)}$$

$$\delta f = f(x+\delta x) - f(x)$$

$$\delta f = \delta x \frac{df}{dx}$$

②

$$f(x+\delta x, y+\delta y) = f(x, y+\delta y) + \delta x \frac{\partial}{\partial x} f(x, y+\delta y)$$

$$= f(x,y) + \delta y \frac{\partial}{\partial y} f(x,y)$$

$$+ \delta x \frac{\partial}{\partial x}\left[f(x,y) + \delta y \frac{\partial}{\partial y} f(x,y)\right]$$

$$= f(x,y) + \delta x \frac{\partial}{\partial x} f + \delta y \frac{\partial}{\partial y} f \quad \text{(upto leading order)}$$

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$$

$$\delta f = \delta x \frac{\partial}{\partial x} f + \delta y \frac{\partial}{\partial y} f$$

③ In 3-D

$$\delta f = \delta x \frac{\partial}{\partial x} f + \delta y \frac{\partial}{\partial y} f + \delta z \frac{\partial}{\partial z} f$$

④ Partial derivative – illustrative example.

$$f(x,y,z) = x^3 y^2 + x \sin y + z^5$$

$$\frac{\partial f}{\partial x} = 3x^2 y^2 + \sin y$$

$$\frac{\partial f}{\partial y} = 2x^3 y + x \cos y$$

$$\frac{\partial f}{\partial z} = 5z^4$$

(5) Total derivative and partial derivative

$$L = L(x(t), t)$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{dx}{dt}\frac{\partial L}{\partial x}$$

In the absence of internal dependence, the partial derivative and total derivative is same.

(6) Variation in vector notation

$$\delta f = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \delta z \frac{\partial f}{\partial z}$$

$$= \delta \vec{r} \cdot \vec{\nabla} f$$

where

$$\delta \vec{r} = \hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(7) <u>Example</u>:

$$r = \sqrt{x^2 + y^2 + z^2}$$

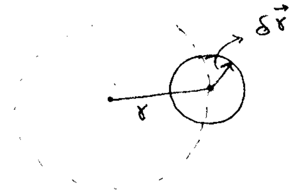
$$\vec{\nabla} r = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\sqrt{x^2+y^2+z^2}$$

$$= \frac{\hat{i}x + \hat{j}y + \hat{k}z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

(8) Interpretation

(i) $\vec{\nabla} f$ is normal to the curve/surface, $f = \text{const}$, in 2-D/3-D.



$$\delta f = \delta\vec{r} \cdot \vec{\nabla} f$$
$$= |\delta\vec{r}|\ |\vec{\nabla} f| \cos\theta$$

(ii) $\vec{\nabla} f$ points in the direction of maximum δf.

(iii) Magnitude of $\vec{\nabla} f$ is the "slope" in the direction of $\delta\vec{r}$ that leads to maximum δf.

9 Examples

(i) $r = \sqrt{x^2 + y^2}$ is the surface of a cylinder.

$\vec{\nabla} r = \hat{r}$ is normal to the cylinder.

(ii) $\theta = \tan^{-1} \frac{y}{x}$ is the surface of a half-plane.

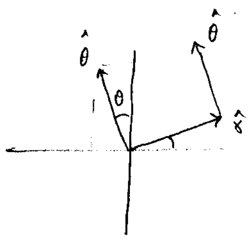
$$\vec{\nabla} \theta = \hat{i} \frac{\partial}{\partial x} \tan^{-1} \frac{y}{x} + \hat{j} \frac{\partial}{\partial y} \tan^{-1} \frac{y}{x} + 0 \hat{k}$$

$$= -\frac{1}{r} \sin\theta \, \hat{i} + \frac{1}{r} \cos\theta \, \hat{j}$$

$\theta = \tan^{-1} \frac{y}{x}$

$\tan\theta = \frac{y}{x}$

$\sec^2\theta \frac{\partial \theta}{\partial x} = -$



$$\hat{r} = \cos\theta \, \hat{i} + \sin\theta \, \hat{j}$$

$$\hat{\theta} = \frac{\vec{\nabla} \theta}{|\vec{\nabla} \theta|} = -\sin\theta \, \hat{i} + \cos\theta \, \hat{j}$$

10 Any system tends to minimize its energy. This dictates the definition of force.

$$\vec{F} = -\vec{\nabla} U$$

11 Examples

(i) Potential energy on surface of Earth

$$U(x,y,z) = mgz$$

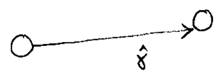
$$\vec{F} = -\vec{\nabla} U$$

$$= -\hat{z}\, mg$$

(ii) Gravitational potential energy

$$U = -\frac{G m_1 m_2}{r}$$

$$U(x,y,z) = -\frac{1}{r}$$



$$\vec{F} = -\frac{\hat{r}}{r^2}$$

(iii) Electrostatic potential energy

$$U = \frac{k q_1 q_2}{r}$$

$$U(x,y,z) = \frac{1}{r}$$

$$\vec{F} = \frac{\hat{r}}{r^2}$$