Date: 2014 Aug 25

Agenda: (i) introduce dyadic notation
(ii) hints and illustrative examples for homework.

(1) Vector = tensor of rank 1.

$$\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i$$

$$= \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

(2) Dyadic = tensor of rank 2

$$\overleftrightarrow{K} = \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{e}_i K_{ij} \hat{e}_j$$

$$= \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

$$\begin{aligned} = \; & \hat{e}_1 K_{11} \hat{e}_1 + \hat{e}_1 K_{12} \hat{e}_2 + \hat{e}_1 K_{13} \hat{e}_3 \\ + \; & \hat{e}_2 K_{21} \hat{e}_1 + \hat{e}_2 K_{22} \hat{e}_2 + \hat{e}_2 K_{23} \hat{e}_3 \\ + \; & \hat{e}_3 K_{31} \hat{e}_1 + \hat{e}_3 K_{32} \hat{e}_2 + \hat{e}_3 K_{33} \hat{e}_3 \end{aligned}$$

(3) For now we will come across dyadics constructed as the 'outer product' of two vectors

$$\overleftrightarrow{K} = \vec{A}\,\vec{B} \equiv \vec{A} \otimes \vec{B}$$

$$= \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \begin{pmatrix} B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} \begin{pmatrix} A_1 B_1 & A_1 B_2 & A_1 B_3 \\ A_2 B_1 & A_2 B_2 & A_2 B_3 \\ A_3 B_1 & A_3 B_2 & A_3 B_3 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

Thus, $K_{ij} = A_i B_j$

(4) Identity in dyadics:

$$\overleftrightarrow{1} = \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

components are $\delta_{ij}$

$$= \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3$$

(5) Caution. Do not interpret a dot product between two vector vector, unless there is one. Three kinds of vector products

(i) $\vec{A}\,\vec{B} \equiv A_i B_j$ → Outer product (tensor of rank 2)

(ii) $\vec{A} \cdot \vec{B} = A_i B_i$ → Dot product (tensor of rank 0)

(iii) $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$ → Cross product (tensor of rank 1)

⑥ $$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

⑦ · Outer product : $$\vec{\nabla}\vec{r} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}x & \frac{\partial}{\partial x}y & \frac{\partial}{\partial x}z \\ \frac{\partial}{\partial y}x & \frac{\partial}{\partial y}y & \frac{\partial}{\partial y}z \\ \frac{\partial}{\partial z}x & \frac{\partial}{\partial z}y & \frac{\partial}{\partial z}z \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

$$= \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$$

Using index notation we would have.

$$\nabla_i x_j = \delta_{ij}$$

(8) Dot product : $\vec{\nabla} \cdot \vec{r} = \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}$

$$= 3$$

Using index notation we will have

$$\nabla_i \, x_i = \delta_{ii} = 3$$

(9) Cross product : $\vec{\nabla} \times \vec{r} = \hat{i} \times \hat{i} + \hat{j} \times \hat{j} + \hat{k} \times \hat{k}$

$$= 0$$

Using index notation we will have

$$\epsilon_{ijk} \nabla_j \, x_k = \epsilon_{ijk} \, \delta_{jk}$$

$$= 0$$

(because of conflict between symmetry and anti-symmetry)

$$\epsilon_{ijk} \, \delta_{jk} = \frac{1}{2} \left[ \epsilon_{ijk} \, \delta_{jk} + \underset{kj}{\epsilon_{ijk}} \, \underset{jk}{\delta_{kj}} \right]$$

$$= \frac{1}{2} \left[ \epsilon_{ijk} \, \delta_{jk} + \epsilon_{ikj} \, \delta_{jk} \right]$$

$$= \frac{1}{2} \left( \epsilon_{ijk} + \epsilon_{ikj} \right) \delta_{jk}$$

$$= \frac{1}{2} \left( \epsilon_{ijk} - \epsilon_{ijk} \right) \delta_{jk}$$

$$= 0 .$$

(10) Recolled the following derivation

$$
\begin{aligned}
\left[\vec{A}\times(\vec{B}\times\vec{C})\right]_i &= \epsilon_{ijk}\, A_j\, \epsilon_{kmn}\, B_m C_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, A_j B_m C_n \\
&= A_j B_i C_j - A_j B_j C_i \\
&= B_i\, A_j C_j - C_i\, A_j B_j \\
&= \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})
\end{aligned}
$$

(assumes A, B, C commute.)

(11)
$$
\begin{aligned}
\left[\vec{\nabla}\times(\vec{B}\times\vec{C})\right]_i &= \epsilon_{ijk}\, \nabla_j\, \epsilon_{kmn}\, B_m C_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, \nabla_j (B_m C_n) \\
&= \nabla_j (B_i C_j) - \nabla_j (B_j C_i) \\
&= (\nabla_j B_i) C_j + B_i (\nabla_j C_j) - (\nabla_j B_j) C_i - B_j (\nabla_j C_i) \\
&= C_j \nabla_j B_i + B_i \nabla_j C_j - C_i \nabla_j B_j - B_j \nabla_j C_i \\
&= \vec{C}\cdot\vec{\nabla}\,\vec{B} + \vec{B}\,\vec{\nabla}\cdot\vec{C} - \vec{C}\,\vec{\nabla}\cdot\vec{B} - \vec{B}\cdot\vec{\nabla}\,\vec{C}
\end{aligned}
$$

(12) Note : $\vec{C}\cdot\underbrace{\vec{\nabla}\,\vec{B}}_{\text{dyadic}} = \text{vector}$

vector

dot product

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$$[\vec{A} \times (\vec{\nabla} \times \vec{c})]_i = \epsilon_{ijk} A_j \epsilon_{kmn} \nabla_m C_n$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j \nabla_m C_n$$
$$= A_j \nabla_i C_j - A_j \nabla_j C_i$$
$$= (\nabla_i C_j) A_j - A_j \nabla_j C_i$$
$$= (\vec{\nabla} \vec{c}) \cdot \vec{A} - \vec{A} \cdot \vec{\nabla} C$$

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$$[\vec{\nabla} \times (\vec{\nabla} \times \vec{c})]_i = \epsilon_{ijk} \nabla_j \epsilon_{kmn} \nabla_m C_n$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \nabla_j \nabla_m C_n$$
$$= \nabla_j \nabla_i C_j - \nabla_j \nabla_j C_i$$
$$= \nabla_i \nabla_j C_j - \nabla_j \nabla_j C_i$$
$$= \vec{\nabla}\, \vec{\nabla} \cdot \vec{C} - \nabla^2 \vec{C}$$