Dali: 2014 Aug 27

1 Gradient

$U = U(x,y,z)$

$$\vec{\nabla} U = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) U$$

$$= \hat{i} \frac{\partial U}{\partial x} + \hat{j} \frac{\partial U}{\partial y} + \hat{k} \frac{\partial U}{\partial z}$$

Interpretation : $\vec{\nabla}$ (Surface) = Normal to Surface.

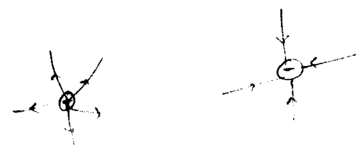
2 Divergence

$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

$A_i = A_i(x,y,z)$

$$\vec{\nabla} \cdot \vec{A} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k} \right)$$

$$= \frac{\partial}{\partial x} A_1 + \frac{\partial}{\partial y} A_2 + \frac{\partial}{\partial z} A_3$$



Interpretation : Source and sink.

$$\vec{\nabla} \cdot \vec{A} \begin{cases} = 0, & \text{if the point is not a source or sink} \\ \neq 0, & \text{otherwise.} \end{cases}$$

3 Curl

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial}{\partial y} A_3 - \frac{\partial}{\partial z} A_2 \right) - \hat{j} \left(\frac{\partial}{\partial x} A_3 - \frac{\partial}{\partial z} A_1 \right) + \hat{k} \left(\frac{\partial}{\partial x} A_2 - \frac{\partial}{\partial y} A_1 \right)$$

Interpretation :
- Vorticity
- Faraday's induction

(4) Laplacian

$$\nabla^2 U = \vec{\nabla} \cdot \vec{\nabla} U$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) U$$

(5) Let us go over examples of vector operations.

(6) $$[\vec{\nabla} \times \vec{\nabla} U]_i = \epsilon_{ijk} \nabla_j \nabla_k U$$
$$= 0$$

$\nabla_j \nabla_k = \nabla_k \nabla_j \rightarrow$ symmetric

(7) $$[(\vec{\nabla} f) \times (\vec{\nabla} g)]_i = \epsilon_{ijk} (\nabla_j f)(\nabla_k g)$$
$$\neq 0$$

$(\nabla_j f)(\nabla_k g) \neq (\nabla_k f)(\nabla_j f)$

(8) $$\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = \nabla_i \epsilon_{ijk} \nabla_j A_k$$
$$= \epsilon_{ijk} \nabla_i \nabla_j A_k$$
$$= 0$$

(9) $$[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})]_i = \epsilon_{ijk} \nabla_j \epsilon_{kmn} \nabla_m A_n$$
$$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \nabla_j \nabla_m A_n$$
$$= \nabla_j \nabla_i A_j - \nabla_j \nabla_j A_i$$
$$= \nabla_i \nabla_j A_j - \nabla_j \nabla_j A_i$$
$$= \vec{\nabla} \, \vec{\nabla} \cdot \vec{A} - \nabla^2 \vec{A}$$

(10)
$$\vec{A} \times (\vec{\nabla} \times \vec{C}) = \epsilon_{ijk} A_j \epsilon_{kmn} \nabla_m C_n$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) A_j \nabla_m C_n$$
$$= A_j \nabla_i C_j - A_j \nabla_j C_i$$
$$= (\nabla_i C_j) A_j - A_j \nabla_j C_i$$
$$= (\vec{\nabla}\, \vec{C}) \cdot \vec{A} - \vec{A} \cdot \vec{\nabla}\, \vec{C}$$

(11) Cylindrical coordinate (similar to homework)

$$\rho = \sqrt{x^2+y^2}$$
$$\phi = \tan^{-1} \frac{y}{x}$$
$$z = z$$

$$\vec{\nabla}\rho = \hat{i}\frac{\partial}{\partial x}\sqrt{x^2+y^2} + \hat{j}\frac{\partial}{\partial y}\sqrt{x^2+y^2} + \hat{k}\frac{\partial}{\partial z}\sqrt{x^2+y^2}$$
$$= \hat{i}\frac{x}{\rho} + \hat{j}\frac{y}{\rho} + \hat{k}\,0$$
$$= \hat{i}\cos\phi + \hat{j}\sin\phi + \hat{k}\,0$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} + \hat{j}\frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} + \hat{k}\frac{\partial}{\partial z}\tan^{-1}\frac{y}{x}$$
$$= -\hat{i}\frac{\sin\phi}{\rho} + \hat{j}\frac{\cos\phi}{\rho} + 0\hat{k}$$

$$\hat{\phi} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = -\hat{i}\sin\phi + \hat{j}\cos\phi + 0\hat{k}$$

$\tan\phi = \frac{y}{x}$

$\sec^2\phi \frac{\partial\phi}{\partial x} = -\frac{y}{x}$

$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{\rho}$

$\frac{\partial\phi}{\partial x} = \frac{\cos\phi}{\rho}$