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δ- function

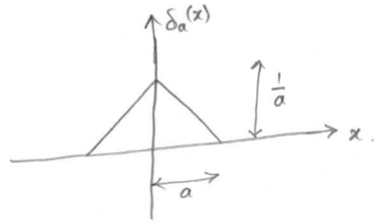
① Definition (that should be interpreted as a limiting case)

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0, \end{cases}$$

and satisfies

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = 1.$$

② Visual illustrative example



$$\delta(x) = \underset{a \to 0}{\mathrm{Lt}}\ \delta_a(x)$$

$$\to \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0, \end{cases}$$

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \text{Area under } \delta_a(x)$$

$$= \frac{1}{2}(2a)\frac{1}{a}$$

$$= 1$$

③ Dimension of δ- function.

$$[\delta(x)] = \frac{1}{[x]}$$

(4) In physical problems bodies or particles are represented by a point. The mass density or charge density of a particle is described by a δ-function. For example, the mass density of a particle in 1-dimension and higher dimensions will be

$$\rho(x,y,z) = m\,\delta(x)\,\delta(y)\,\delta(z) \qquad \lambda = \frac{\text{mass}}{\text{length}}$$

$$\rho(x,y,z) = \lambda\,\delta(x)\,\delta(y)$$

$$\rho(x,y,z) = \sigma\,\delta(z) \qquad \sigma = \frac{\text{mass}}{\text{area}}.$$

Notice that

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\; \rho(x,y,z) = m$$

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\; \rho(x,y,z) = \lambda$$

$$\int_{-\infty}^{+\infty} dz\; \rho(x,y,z) = \sigma$$

confirm with total mass in the corresponding spaces.

(5) Replacing $m \to q$ leads to the respective charge densities that we will encounter in this course.

6 A particular representation for δ-function is

$$\delta(x) = \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \; Im \left(\frac{1}{x - i\epsilon} \right) \qquad \frac{1}{x - i\epsilon} = \frac{x + i\epsilon}{x^2 + \epsilon^2}$$

$$= \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \frac{\epsilon}{x^2 + \epsilon^2}$$

For any $x \neq 0$ we can always find $\epsilon \ll x$, thus,

$$\delta(x) = \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \frac{\epsilon}{x^2} \to 0, \qquad (x \neq 0).$$

For $x = 0$

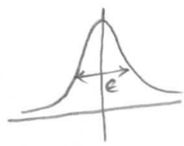
$$\delta(x) = \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \frac{1}{\epsilon} \to \infty, \qquad x = 0.$$

Further

$$\int_{-\infty}^{+\infty} dx \; \delta(x) = \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \int_{-\infty}^{+\infty} dx \frac{\epsilon}{x^2 + \epsilon^2}. \qquad \begin{aligned} x &= \epsilon \tan\theta \\ dx &= \epsilon \sec^2\theta \, d\theta \end{aligned}$$

$$= \frac{1}{\pi} \underset{\epsilon \to 0+}{Lt} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cancel{\epsilon} \cancel{\sec^2}\theta \, d\theta \; \cancel{\epsilon}}{\cancel{\epsilon^2} \cancel{\sec^2}\theta}$$

$$= 1.$$



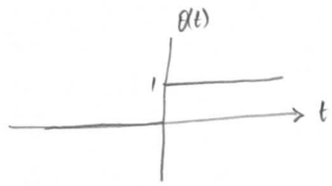
The function

$$\frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}.$$

is the Cauchy distribution, which is also known as the Lorentz distribution.

7 δ-function is the derivative of the Heaviside step functn (θ-functn)

$$\theta(t) = \begin{cases} 1, & \text{if } t>0, \\ 0, & \text{if } t<0. \end{cases} = \underset{\epsilon\to 0+}{Lt}\; i\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega t}}{\omega+i\epsilon}$$



The integral representation for θ(t) can be verified using Cauchy's theorem in complex analysis.

$$\delta(t) = \frac{d}{dt}\,\theta(t)$$

$$= \frac{d}{dt}\,\underset{\epsilon\to 0+}{Lt}\; i\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega t}}{\omega+i\epsilon}$$

$$e^{-i\omega t} \to e^{-i(\omega+i\epsilon)t} = e^{-i\omega t+\epsilon t}$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t},$$

which is an integral representation for δ-function. Incidentally, we learn that the Fourier transform of a δ-function is 1.

(8) Yet another representation for δ-function is.

$$\delta(x) = \frac{1}{\sqrt{\pi}} \operatorname*{Lt}_{\sigma \to 0} \frac{1}{\sigma} e^{-\frac{x^2}{\sigma^2}} .$$

For $x \neq 0$,

$$\delta(x) = \frac{1}{\sqrt{\pi}} \operatorname*{Lt}_{\sigma \to 0} \frac{1}{\sigma} e^{-\frac{x^2}{\sigma^2}} = 0 ,$$

because the exponential beats the $\frac{1}{\sigma}$.

For $x = 0$,

$$\delta(x) = \frac{1}{\sqrt{\pi}} \operatorname*{Lt}_{\sigma \to 0} \frac{1}{\sigma} \to \infty .$$

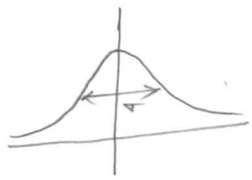
Further

$$\int_{-\infty}^{+\infty} dx\, \delta(x) = \frac{1}{\sqrt{\pi}} \operatorname*{Lt}_{\sigma \to 0} \frac{1}{\sigma} \int_{-\infty}^{+\infty} dx\, e^{-\frac{x^2}{\sigma^2}}$$

$$= 1 .$$

The function

$$\frac{1}{\sqrt{\pi}} \frac{1}{\sigma} e^{-\frac{x^2}{\sigma^2}}$$



is the Gaussian distribution, and σ is the standard deviation of the Gaussian distribution.

is the

(9) In practice the δ-function is used in the following form,

$$\int_{-\infty}^{+\infty} dx\ \delta(x)\ f(x) = f(0).$$

(10) An immediate generalizatn is

$$\int_{-\infty}^{+\infty} dx\ \delta(x-a)\ f(x) = f(a).$$

(11) Examples:

(i) $$\int_{-\infty}^{+\infty} dx\ \delta(x)\ [x^2 + 2x + 3] = 3$$

(ii) $$\int_{-\infty}^{+\infty} dx\ \delta(x-5)\ [x^2 + 2x + 3] = 38$$

(iii) $$\int_{-\infty}^{+\infty} dx\ \delta(x+5)\ [x^2 + 2x + 3] = 18$$

(12) Examples:

(i) $$\int_{1}^{\infty} dx\ \delta(x)\ [x^2 + 2x + 3] = 0$$

(ii) $$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{\epsilon}^{\infty} dx\ \delta(x)\ [x^2 + 2x + 3] = 0$$

(iii) $$\underset{\epsilon \to 0}{\mathrm{Lt}} \int_{-\epsilon}^{\epsilon} dx\ \delta(x)\ [x^2 + 2x + 3] = 3$$

(13) We shall later in the course have the opportunity to define

$$\delta(f(x))$$

which we will show gets contributions at the roots of $f(x) = 0$. For now we will confine our attention to linear functions of the form

$$f(x) = ax + b.$$

In particular,

$$\delta(ax+b) = \frac{1}{|a|}\,\delta\left(x + \frac{b}{a}\right), \qquad a \neq 0.$$

If $b=0$, we have

$$\delta(ax) = \frac{1}{|a|}\,\delta(x), \qquad a \neq 0.$$

(14) Proof:

$\underline{\underline{a > 0}}$

$$\int_{-\infty}^{+\infty} dx\; \delta(ax+b)\, f(x) = \int_{-\infty}^{+\infty} \frac{dy}{a}\, \delta(y+b)\, f\left(\frac{y}{a}\right) \qquad y = ax.$$

$$= \frac{1}{a} f\left(-\frac{b}{a}\right)$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} dx\; \delta\left(x + \frac{b}{a}\right) f(x)$$

$$\Rightarrow \quad \delta(ax+b) = \frac{1}{a}\,\delta\left(x+\frac{b}{a}\right), \qquad a > 0.$$

(15) For $a < 0$,

$$y = ax$$
at $x = -\infty$, $y = +\infty$
at $x = +\infty$, $y = -\infty$
$$dy = a\,dx$$

$$\int_{-\infty}^{+\infty} dx \; \delta(ax+b)\, f(x) = \int_{+\infty}^{-\infty} \frac{dy}{a} \; \delta(y+b)\, f\left(\frac{y}{a}\right)$$

$$= -\frac{1}{a} \int_{-\infty}^{+\infty} dy \; \delta(y+b)\, f\left(\frac{y}{a}\right)$$

$$= -\frac{1}{a} f\left(-\frac{b}{a}\right)$$

$$= -\frac{1}{a} \int_{-\infty}^{+\infty} dx \; \delta\left(x + \frac{b}{a}\right) f(x)$$

$$\Rightarrow \quad \delta(ax+b) = -\frac{1}{a}\,\delta\left(x+\frac{b}{a}\right), \quad a < 0.$$

In general,

$$\delta(ax+b) = \frac{1}{|a|}\,\delta\left(x + \frac{b}{a}\right)$$

(16) Examples:

(i) $$\int_{-\infty}^{+\infty} dx \; \delta(2x+1)\left[16x^2 + 2x + 1\right] = \frac{1}{2}\left[16\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) + 1\right] = 2$$

(ii) $$\int_{-\infty}^{+\infty} dx \; \delta(2x-1)\left[16x^2 + 2x + 1\right] = \frac{1}{2}\left[16\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 1\right] = 3$$

(iii) $$\int_{-\infty}^{+\infty} dx \; \delta(-2x+1)\left[16x^2 + 2x + 1\right] = \frac{1}{2}\left[16\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 1\right] = 3$$

(iv) $$\int_{-\infty}^{+\infty} dx \; \delta(-2x-1)\left[16x^2 + 2x + 1\right] = \frac{1}{2}\left[16\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)^2 + 1\right] = 2$$

17 Let us evaluate

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \frac{3}{r^3} - \frac{3}{r^3} = 0 .$$

Is this true everywhere? What about at $r = 0$?

Consider

V – sphere of radius R.

$$\int_V d^3r \; \vec{\nabla} \cdot \frac{\vec{r}}{r^3} = \oint d\vec{a} \cdot \frac{\vec{r}}{r^3}$$

$$= 4\pi$$

How can the integral of a quantity that is zero everywhere lead to a non-zero answer?

Thus, we write

$$\int_V d^3r \; \vec{\nabla} \cdot \frac{\vec{r}}{r^3} = 4\pi = 4\pi \int d^3r \; \delta^{(3)}(\vec{r})$$

and read out

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^3} = 4\pi \; \delta^{(3)}(\vec{r}) ,$$

which is valid at $\vec{r} = 0$ too.