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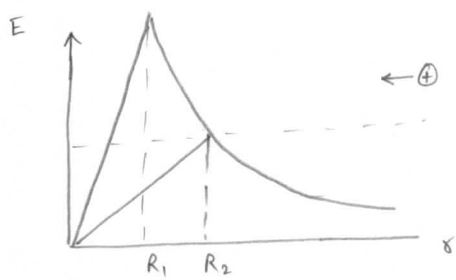
① Rutherford model of atom

$$\vec{E} = \begin{cases} \dfrac{kQ}{R^2}\dfrac{r}{R}\,\hat{r} & r < R \\[2ex] \dfrac{kQ}{r^2}\,\hat{r} & R < r \end{cases}$$



② Electric field of a hydrogen atom.

Quantum mechanically the probability density of finding an electron around the proton is

$$P(r) = \frac{1}{8\pi}\left(\frac{2}{a_0}\right)^3 e^{-\frac{2r}{a_0}}$$

where a_0 is the Bohr radius

$$a_0 = \frac{\hbar}{m_e c\,\alpha} \sim 5 \times 10^{-11}\ \text{m}$$

Note that

$$\int d^3r\ P(r) = 4\pi \frac{1}{8\pi}\left(\frac{2}{a_0}\right)^3 \int_0^\infty r^2 dr\, e^{-\frac{2r}{a_0}}$$

$$= \frac{1}{2}\int_0^\infty \frac{dt}{t}\, t^3 e^{-t} = 1 \quad ✓$$

3 Let us calculate the electric field in and around the hydrogen atom. (Assume proton is a point particle.)

$\rho(r) = -e\, P(r)$ ↳ for electron

$$E \, 4\pi R^2 = \frac{Q_{in}}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{1}{R^2} Q_{in}$$

$$Q_{in} = +e - e \int_R d^3r \, P(r)$$

$$= e\left[1 - 4\pi \int_0^R r^2 dr \, \frac{1}{8\pi} \left(\frac{2}{a_0}\right)^3 e^{-\frac{2r}{a_0}}\right]$$

$$= e\left[1 - \frac{1}{2} \int_0^{\frac{2R}{a_0}} \frac{dt}{t} \, t^3 e^{-t}\right]$$

$$= e \, \frac{1}{2} \int_{\frac{2R}{a_0}}^{\infty} \frac{dt}{t} \, t^3 e^{-t}$$

$$= e \, e^{-\frac{2R}{a_0}} \left[1 + \frac{2R}{a_0} + \frac{1}{2}\left(\frac{2R}{a_0}\right)^2\right]$$

$$\int_x^\infty dt \, t^2 e^{-t}$$
$$= x^2 - \int_x^\infty dt \, 2t \frac{d}{dt} e^{-t}$$
$$= x^2 e^{-x} + 2x e^{-x} + 2e^{-x}$$
$$= e^{-x}\left[x^2 + 2x + 2\right]$$

Thus,

$$E = \frac{1}{4\pi\epsilon_0} \frac{e}{R^2} \left[1 - \frac{1}{6}\left(\frac{2R}{r_0}\right)^3 + O\left(\left(\frac{2R}{r_0}\right)^4\right)\right] \quad \text{for } 2R < a_0$$

$$\frac{1}{4\pi\epsilon_0} \frac{e}{a_0^2} \left[1 + \frac{2R}{a_0} + \frac{1}{2}\left(\frac{2R}{a_0}\right)^2\right] e^{-\frac{2R}{a_0}} \quad \text{for } a_0 < 2R.$$

4 Example

Solid sphere with $\rho(r) = A r^n$

$$Q = \int d^3r \, \rho(r) = 4\pi \int_0^R r^2 dr \, A r^n = 4\pi A \frac{R^{n+3}}{(n+3)}$$

$$A = \frac{(n+3)\, Q}{4\pi R^{n+3}}$$

Using Gauss's law

$$E \, 4\pi r^2 = \frac{Q_{in}}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} Q_{in}$$

for $r < R$:

$$Q_{in} = 4\pi \int_0^r r'^2 dr' \, A r'^n$$

$$= 4\pi A \frac{r^{n+3}}{n+3}$$

$$= \cancel{4\pi} \frac{\cancel{(n+3)}\, Q}{\cancel{4\pi} R^{n+3}} \frac{r^{n+3}}{\cancel{n+3}}$$

$$= Q \left(\frac{r}{R}\right)^{n+3}$$

Thus,

$$E = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2} \left(\dfrac{r}{R}\right)^{n+3}, & r < R, \\[2ex] \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2}, & R < r. \end{cases}$$

This leads to the earlier evaluated expression for $n = 0$ (uniform charge distribution).