Date : 2014 Sep 22.

1. Electrostatics

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \frac{\partial \rho}{\partial t} = 0 \qquad \vec{F} = q\vec{E}.$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad \frac{\partial \vec{E}}{\partial t} = 0$$

2. We discussed the Gauss's law

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \iff \oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{in}}{\epsilon_0}$$

3. We next study the content of the curl of electric field being zero. We will see that

$$\left.\begin{aligned} &\vec{\nabla}\times\vec{E} = 0 \\ \Rightarrow\ &\vec{E} = -\vec{\nabla}\phi \end{aligned}\right\} \iff -\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{E} = \phi(\vec{b}) - \phi(\vec{a}).$$

(4) Let us recollect, or discuss, conservative forces, which are forces that can be written in terms of potential energy. Further, the change in potential energy between two points, in the case of conservative forces, is independent of the path taken. Thus, in summary we have the following:

(i) A conservative force can be completely expressed in terms of a potential energy function, i.e.

$$\vec{F} = - \vec{\nabla} U$$

(ii) The curl of a conservative force is zero, if $\vec{F}$ is conservative.

$$\vec{\nabla} \times \vec{F} = 0 ,$$

(iii) The change in potential energy between points $\vec{a}$ and $\vec{b}$, which is the work needed to move a body from point $\vec{a}$ to point $\vec{b}$, is independent of the path taken by the body, i.e.

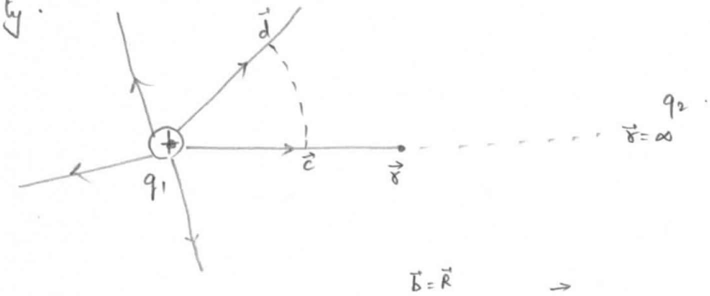
$$U(\vec{b}) - U(\vec{a}) = - \int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{F} .$$

(5) Intuitive examples:

→ Gravitational force between two masses is conservative.

→ Frictional force is not conservative.

(6) Work done to move a positive charge from infinity.



$$d\vec{l} = dl(-\hat{r}) = (-dr)(-\hat{r}) = dr\,\hat{r}$$

$$U(\vec{b}=\vec{R}) - U(\vec{a}=\infty) = -\int_{\vec{a}=\infty}^{\vec{b}=\vec{R}} d\vec{l} \cdot \vec{F}$$

$$= -\int_{\vec{a}=\infty}^{\vec{b}=\vec{R}} dr\,\hat{r} \cdot \frac{k q_1 q_2}{r^2}\,\hat{r}$$

$$= -\int_{r=\infty}^{r=R} dr \frac{k q_1 q_2}{r^2}$$

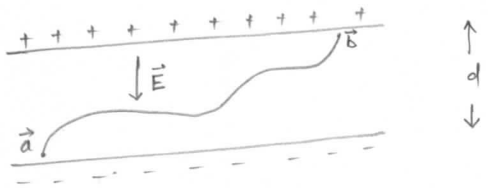
$$= \frac{k q_1 q_2}{R}$$

(7) Note that

$$U(\vec{c}) - U(\vec{d}) = 0$$

because the path is perpendicular to the electric field. This leads to the concept of equipotential surfaces.

(8) Consider the example of electric field between two plates with surface charge density σ.



$$E = \frac{\sigma}{\epsilon_0}$$

→ Recollect that the electric field is a constant between the plates. $\vec{E} = \frac{\sigma}{\epsilon_0}(-\hat{z})$ $\vec{F} = q\vec{E}$

→ $$U(\vec{b}) - U(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{F}$$

$$= -\int_{\vec{a}}^{\vec{b}} (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) \cdot q\frac{\sigma}{\epsilon_0}(-\hat{k})$$

$$= q\frac{\sigma}{\epsilon_0} d$$

(9) Next consider a line charge.

$$\vec{E} = k\frac{2\lambda}{\rho}\hat{\rho}$$

$$d\vec{l} = d\rho\,\hat{\rho} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

$$U(\vec{b}) - U(\vec{a}=\infty) = -\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot q\vec{E}$$

$$= -q\int_{\rho=a}^{\rho=b} d\rho\,\frac{k\,2\lambda}{\rho}$$

$$= -q\,2k\lambda \ln\frac{b}{a}$$

(10) Electric potential

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\vec{F} \arrow[r, leftrightarrow, "\vec{F} = q\vec{E}"] \arrow[d, leftrightarrow, "\Delta U = -\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{F}"'] & \vec{E} \arrow[d, leftrightarrow, "\Delta\phi = -\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{E}"] \\
\Delta U \arrow[r, leftrightarrow, "\Delta U = q\,\Delta\phi"'] & \Delta\phi
\end{tikzcd}
$$

(11) Using (6) to (9) we can derive the potential difference between the following surfaces to be:

(i) Parallel plates with surface charge $\sigma = \frac{Q}{A}$ $\qquad V = \dfrac{Q}{\left(\frac{\epsilon_0 A}{d}\right)}$

(ii) Concentric cylinders with surface charge. $\lambda = \frac{Q}{L}$ $\qquad V = \dfrac{Q}{\left(\frac{2\pi\epsilon_0 L}{\ln\frac{b}{a}}\right)}$

(iii) Concentric spheres with surface charge $\sigma = \frac{Q}{A}$ $\qquad V = \dfrac{Q}{4\pi\epsilon_0\left(\frac{ab}{b-a}\right)}$

(12) Capacitance:

Consider two conductors of arbitrary shape

$$V = \frac{Q}{C}$$

(13) Parallel plate: $C = \frac{\epsilon_0 A}{d}$

Concentric cylinder: $C = 2\pi\epsilon_0 L \ln\frac{b}{a}$

Concentric spheres: $C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right)$