

Date: 2014 Sep 29

① Maxwell's equations (Electrostatics)

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} \phi$$

Together we have

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho$$

→ Poisson equation

② For a point charge we have

$$\rho(\vec{r}, t) = q \delta^{(3)}(\vec{r} - \vec{r}_0(t))$$

↘ position of charge.

③ Thus, the Poisson equation for a point charge reads

$$-\nabla^2 \phi = \frac{q}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}_0(t))$$

④ To keep the illustration transparent, let us consider a stationary point charge at the origin,

$$\vec{r}_0(t) = 0.$$

Thus,

$$-\nabla^2 \phi = \frac{q}{\epsilon_0} \delta^{(3)}(\vec{r}).$$

⑤ Fourier transformation :

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \tilde{\phi}(\vec{k})$$

$$\tilde{\phi}(\vec{k}) = \int d^3r e^{-i\vec{k} \cdot \vec{r}} \phi(\vec{r})$$

$$\delta^{(3)}(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}}$$

⑥ Using ⑤ in ④ we have.

$$k^2 \tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0}$$

$$\tilde{\phi}(\vec{k}) = \frac{q}{\epsilon_0} \frac{1}{k^2}$$

⑦ Using ⑥ in ⑤

$$\phi(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \frac{q}{\epsilon_0} \frac{1}{k^2}$$

$$= \frac{1}{8\pi^3} 2\pi \frac{q}{\epsilon_0} \int_0^\infty k^2 dk \frac{1}{k^2} \int_0^\pi \sin\theta d\theta e^{ikr \cos\theta}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\pi} \int_0^\infty dk \int_{-1}^1 dt e^{ikrt}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\pi} \int_0^\infty dk \frac{2}{kr} \sin kr$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r} \frac{2}{\pi} \underbrace{\int_0^\infty \frac{dx}{x} \sin x}_{= \frac{\pi}{2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r} \quad \checkmark$$

choose $\vec{r}$ along $\hat{z}$.

$$\begin{aligned} & \int_{-1}^1 dt e^{ikrt} \\ &= \frac{1}{ikr} [e^{ikr} - e^{-ikr}] \\ &= \frac{2}{kr} \sin kr \end{aligned}$$