Date: 2014 Oct 1

1. We have derived the solution to the Poisson equation for a single point charge

$$-\nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} q\, \delta^{(3)}(\vec{r}-\vec{r}_0)$$

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{r}_0|}$$

2. We define the Green's function as the solution to the differential equation

$$-\nabla^2 G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\pi} \frac{1}{|\vec{r}-\vec{r}\,'|}$$

Thus, the Green's function is the electric potential due to a charge with $q = \epsilon_0$!

3. Symbolically, we can write

$$-\nabla^2 G = 1$$

$$G = \frac{1}{(-\nabla^2)}$$

Thus, Green's function is the inverse of the negative of the Laplacian operator.

4 Now, consider the Poisson equation with an arbitrary charge density $\rho(\vec{r})$ as the source:

$$- \nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r})$$

5 Multiply by the Green's function and integrate

$$- G(\vec{r}', \vec{r}) \, \nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} G(\vec{r}', \vec{r}) \, \rho(\vec{r})$$

$$- \int d^3r \, G(\vec{r}', \vec{r}) \, \nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} \int d^3r \, G(\vec{r}', \vec{r}) \, \rho(\vec{r})$$

6 Integrate by parts, twice, and each time we the boundary conditions on $\phi(\vec{r})$ that they go to zero at infinity, to obtain

$$- \int d^3r \left[\nabla^2 G(\vec{r}', \vec{r}) \right] \phi(\vec{r}) = \frac{1}{\epsilon_0} \int d^3r \, G(\vec{r}', \vec{r}) \, \rho(\vec{r})$$

7 $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$ for the Green's function in 2. But This is true for the we will later see it to be generically true.

8 Using the definition of Greens function, see ②, in equatn ⑥ we have.

$$\int d^3r \; \delta^{(3)}(\vec{r}\,'-\vec{r}) \; \phi(\vec{r}) = \frac{1}{\epsilon_0} \int d^3r \; G(\vec{r}\,', \vec{r}) \; \rho(\vec{r})$$

$$\phi(\vec{r}\,') = \frac{1}{\epsilon_0} \int d^3r \; G(\vec{r}\,', \vec{r}) \; \rho(\vec{r})$$

$\vec{r}\,' \leftrightarrow \vec{r}$

(OR)
$$\phi(\vec{r}) = \frac{1}{\epsilon_0} \int d^3r' \; G(\vec{r}, \vec{r}\,') \; \rho(\vec{r}\,')$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \; \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

This is the statement of superposition principle in electrostatics!

9 Consider a point dipole

$$\rho(\vec{r}) = \underset{\substack{|\vec{a}|\to 0 \\ q \to 0 \\ 2aq \to \text{fixed.}}}{\mathrm{Lt}} \left[q \; \delta^{(3)}(\vec{r} - \vec{a}) - q \; \delta^{(3)}(\vec{r} + \vec{a}) \right]$$

i.e. $a \to 0$, $q \to \infty$ keeping $p = 2aq = \text{fixed}$

(10) Let the charges be on the z-axis.

$$\rho(\vec{r}) = \underset{(\text{point-dipole})}{Lt} \; 2aq \; \delta(x)\,\delta(y) \left[\frac{\delta(z-a) - \delta(z+a)}{2a} \right]$$

$$= - \underset{\substack{a\to 0 \\ q \to 0 \\ 2aq \to \text{fixed}}}{Lt} \; p \; \delta(x)\,\delta(y)\, \frac{d}{dz}\delta(z)$$

$p = 2aq.$

(11) More generally we have.

$$\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r})$$

(12) Let us now determine the potential due to a dipole:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho_{dipole}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{1}{|\vec{r}-\vec{r}'|}\left(-\vec{p}\cdot\vec{\nabla}'\right)\delta^{(3)}(\vec{r}')$$

$\downarrow$ integrate by parts, surface term does not contribute.

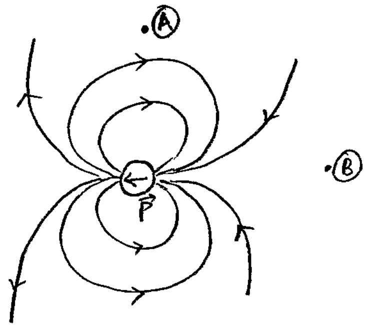
$$= \frac{1}{4\pi\epsilon_0}\int d^3r' \left[\left(\vec{p}\cdot\vec{\nabla}'\right)\frac{1}{|\vec{r}-\vec{r}'|} \right] \delta^{(3)}(\vec{r}')$$

also use $\frac{\partial}{\partial x} f(x-x') = -\frac{\partial}{\partial x'} f(x-x')$

$$= -\frac{1}{4\pi\epsilon_0}\,\vec{p}\cdot\vec{\nabla}\,\frac{1}{r} = \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\vec{r}}{r^3}$$

(13)
$$\vec{E} = -\vec{\nabla}\phi$$
$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$
$$= -\frac{1}{4\pi\epsilon_0}\left[\{\underbrace{\vec{\nabla}(\vec{r}\cdot\vec{p})}_{=\overleftrightarrow{1}}\}\frac{1}{r^3} + (\vec{p}\cdot\vec{r})\underbrace{\left(\vec{\nabla}\frac{1}{r^3}\right)}_{=-3\frac{1}{r^4}\hat{r}}\right]$$
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

(14)



(15) At point Ⓐ : $\vec{p}\cdot\hat{r} = 0$
$$\Rightarrow \quad \vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}}{r^3}$$

At point Ⓑ : $\vec{p}\cdot\hat{r} = \vec{p}$ and $\hat{r} = \hat{p}$
$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{r^3}.$$