Date: 2014 Oct 3

① Maxwell's equation for electrostatics is

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho \\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}\right\} \Rightarrow \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\Rightarrow -\nabla^2\phi = \frac{1}{\epsilon_0}\rho \qquad \Rightarrow \vec{E} = -\vec{\nabla}\phi$$

② Let us illustrate this for a uniformly charged solid sphere, whose electric field can be calculated using Gauss's law exactly.

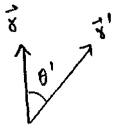
③ The charge density for a uniformly charged solid sphere is

$$\rho_0 = \frac{Q}{\left(\frac{4}{3}\pi R^3\right)}$$

$$\rho(\vec{r}) = \rho_0\,\theta(R-r) \qquad \theta(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases}$$

④ Thus, the electric potential for a solid sphere is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho_0\,\theta(R-r')}{|\vec{r}-\vec{r}'|}$$



$$= \frac{\rho_0}{4\pi\epsilon_0}\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \frac{\theta(R-r')}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}$$

$$= \frac{\rho_0}{2\epsilon_0}\int_0^R r'^2 dr' \underbrace{\int_0^\pi \sin\theta' d\theta' \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}}_{I(r,r')}$$

5 Substitute $\cos\theta' = t$

$-\sin\theta' d\theta' = dt$

$r^2 + r'^2 - 2rr't = y$

$-2rr'dt = dy$

$$I(r, r') = \int_{-1}^{+1} dt \, \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}$$

$$= \frac{1}{2rr'} \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

$$\int \frac{dy}{\sqrt{y}} = 2\sqrt{y}$$

$$= \frac{1}{2rr'} \, 2\left[r + r' - |r - r'| \right]$$

$$= \begin{cases} \dfrac{2}{r'}, & r < r' \\ \dfrac{2}{r} & r' < r \end{cases} = \frac{2}{\mathrm{Max}(r, r')} = \frac{2}{r_>}$$

6 Using 5 in 4

$$\phi(r) = \frac{\rho_0}{2\epsilon_0} \int_0^R r'^2 dr' \, \frac{2}{\mathrm{Max}(r, r')}$$

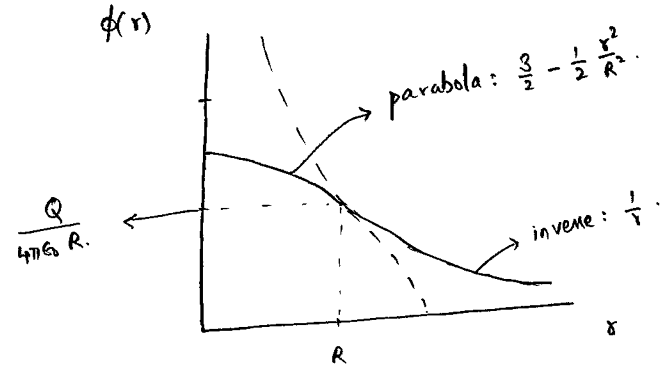
$$= \begin{cases} \dfrac{\rho_0}{\epsilon_0} \displaystyle\int_0^r r'^2 dr' \, \frac{1}{r} + \frac{\rho}{\epsilon_0} \int_r^R r'^2 dr' \, \frac{1}{r'}, & r < R \\ \dfrac{\rho_0}{\epsilon_0} \displaystyle\int_0^R r'^2 dr' \, \frac{1}{r}, & R < r \end{cases}$$

$$= \begin{cases} \dfrac{\rho_0}{\epsilon_0} \left(\dfrac{r^2}{3} + \dfrac{R^2}{2} - \dfrac{r^2}{2} \right), & r < R \\ \dfrac{\rho_0}{\epsilon_0} \dfrac{R^3}{3r}, & R < r. \end{cases}$$

$$\rho_0 = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$= \begin{cases} \dfrac{Q}{4\pi\epsilon_0} \dfrac{1}{R} \left(\dfrac{3}{2} - \dfrac{1}{2} \dfrac{r^2}{R^2} \right), & r < R, \\ \dfrac{Q}{4\pi\epsilon_0} \dfrac{1}{r}, & R < r. \end{cases}$$

(7)



(8)

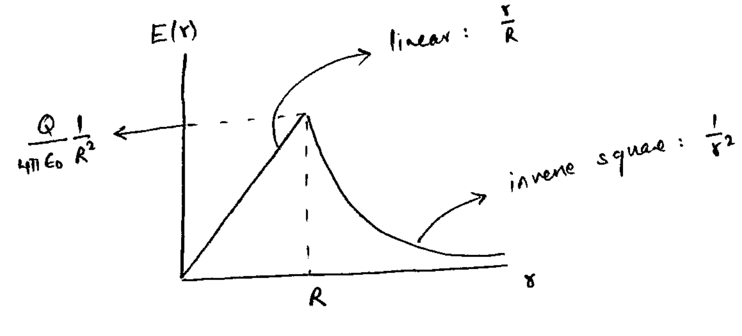
$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\vec{\nabla}\begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{1}{R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), & r < R, \\ \frac{Q}{4\pi\epsilon_0}\frac{1}{r}, & R < r. \end{cases}$$

$$\vec{\nabla} r^2 = 2r\,\vec{\nabla} r = 2r\,\hat{r}$$

$$\vec{\nabla}\frac{1}{r} = -\frac{1}{r^2}\vec{\nabla} r = -\frac{\hat{r}}{r^2}$$

$$= \begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{r}{R}, & r < R, \\ \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}, & R < r. \end{cases}$$



(9) Note that

$$E(r) = -\frac{\partial}{\partial r}\phi(r)$$

graphically.