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Uniqueness of solutions in electrostatics

1. Maxwell's equations for electrostatics are

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \text{and} \qquad \frac{\partial\rho}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi \qquad \frac{\partial\vec{E}}{\partial t} = 0$$

2. Together they read

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\rho$$

What about boundary conditions?

3. We shall show that if $\vec{E} = 0$ at $\vec{r}\to\infty$, it is a sufficient condition for unique solution for $\vec{E}$, using ②.

4. We shall prove this by contradiction. Let $\vec{E}_1$ and $\vec{E}_2$ be distinct (thus non-unique) solutions,

$$\vec{\nabla}\cdot\vec{E}_1 = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{E}_2 = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{E}_1 = 0 \qquad \vec{\nabla}\times\vec{E}_2 = 0$$

5. Subtracting the equations in ④ we have

$$\vec{\nabla}\cdot(\vec{E}_1 - \vec{E}_2) = 0$$

$$\vec{\nabla}\times(\vec{E}_1 - \vec{E}_2) = 0$$

6 Let us define

$$\vec{\mathcal{E}} = \vec{E}_1 - E_2 .$$

Thus, if the solutions are unique, we should be able to show that $\vec{\mathcal{E}} = 0$ everywhere.

7 $$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathcal{E}}) = \vec{\nabla}\, \vec{\nabla} \cdot \vec{\mathcal{E}} - \nabla^2 \vec{\mathcal{E}}$$

Since $\vec{\nabla} \cdot \vec{\mathcal{E}} = 0$ and $\vec{\nabla} \times \vec{\mathcal{E}} = 0$ everywhere. we have. its derivatives also to be zero. Thus,

$$-\nabla^2 \vec{\mathcal{E}} = 0 .$$

8 Let us investigate a single component of the above vector equation.

$$-\nabla^2 \mathcal{E}_x = 0 .$$

9 Since we began with the assumption $\vec{\mathcal{E}} \neq 0$ we can multiply it as

$$-\mathcal{E}_x \nabla^2 \mathcal{E}_x = 0$$

$$\Rightarrow \quad -\vec{\nabla} \cdot (\mathcal{E}_x \vec{\nabla} \mathcal{E}_x) + (\vec{\nabla} \mathcal{E}_x)^2 = 0$$

$$\Rightarrow \quad (\vec{\nabla} \mathcal{E}_x)^2 - \nabla^2 \frac{1}{2} \mathcal{E}_x^2 = 0$$

(10) We integrate the above equation over the volume of a sphere of radius R.

$$\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - \int_V d^3r\,\nabla^2\left(\tfrac{1}{2}\mathcal{E}_x^2\right) = 0$$

$$\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - \int_S d\vec{a}\cdot\vec{\nabla}\left(\tfrac{1}{2}\mathcal{E}_x^2\right) = 0 \qquad \text{using divergence theorem}$$

$$\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - \int_S da\; R^2\frac{\partial}{\partial R}\left(\tfrac{1}{2}\mathcal{E}_x^2\right) = 0 \qquad \hat{r}\cdot\vec{\nabla} = \frac{\partial}{\partial r}$$

$$\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - R^2\frac{\partial}{\partial R}\,\tfrac{1}{2}\langle\mathcal{E}_x^2\rangle = 0 \qquad \langle\mathcal{E}_x^2\rangle = \int_S da\;\mathcal{E}_x^2$$

(11) Divide out by $4\pi R^2$. It can be shown (see Schwinger et. al.) that $R=0$ does not t issues.

$$\frac{1}{R^2}\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - \frac{\partial}{\partial R}\,\tfrac{1}{2}\langle\mathcal{E}_x^2\rangle = 0$$

(12) Integrate over R.

$$\int_0^\infty dR\,\frac{1}{R^2}\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2 - \int_0^\infty dR\,\frac{\partial}{\partial R}\,\tfrac{1}{2}\langle\mathcal{E}_x^2\rangle = 0$$

$$\underbrace{\int_0^\infty dR\,\frac{1}{R^2}\int_V d^3r\,(\vec{\nabla}\mathcal{E}_x)^2}_{\text{positive}} - \underbrace{\tfrac{1}{2}\langle\mathcal{E}_x^2\rangle_\infty}_{=0 \text{ using boundary condition}} + \underbrace{\tfrac{1}{2}\langle\mathcal{E}_x^2\rangle_{R=0}}_{\text{positive}} = 0$$

(13) After using the boundary condition, $\mathcal{E}_x = 0$ at $R \to \infty$, we have sum of two positive terms adding to zero. This is possible only if we require

(i) $\mathcal{E}_x = 0$, at $R = 0$.

(ii) $\vec{\nabla} \mathcal{E}_x = 0$, everywhere.

(14) Since, our choice of origin of sphere in (10) was arbitrary, we further have

$$\mathcal{E}_x = 0, \quad \text{everywhere.}$$

(15) Thus, we have contradicted our assumption

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_1 - \vec{\mathcal{E}}_2 = 0,$$

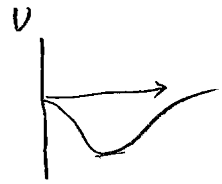
which implies that the solutions to electrostatic configuration are unique.

Earnshaw's theorem

(16) Earnshaw's theorem states that stable configurations are impossible in electrostatics.

(17) Remember,

$$q\phi = U$$
$$q\vec{E} = \vec{F}$$



(18) Requirements of stability

(i) $\vec{F} = 0$ (OR) (i) $\vec{E} = 0$

(ii) $\vec{\nabla}\cdot\vec{F} > 0$ (ii) $\vec{\nabla}\cdot\vec{E} > 0$

(19) Since the test charge can not occupy the point where the source site, we will always have, at the site of test charge $\vec{\nabla}\cdot\vec{E} = 0$.

(20) Comments:

→ One should analyse starting from interaction energy.
→ Quartic potentials will satisfy both conditions.
→ Levitron - spinning top (not statics)
→ Levitating frog (diamagnetism)
→ Levitating magnets near surfaces (presence of boundaries)