Date : 2014 Oct 8

① Let us consider the Poisson equation for an infinitely thin planar sheet with a uniform charge density σ,

$\sigma = \frac{\text{charge}}{\text{Area}}$

$$\rho(\vec{r}) = \sigma \, \delta(z-a)$$

a – position of plate.

② Thus, the Poisson equation, for the electric potential, reads

$$-\nabla^2 \phi(\vec{r}) = \frac{1}{\epsilon_0} \sigma \, \delta(z-a)$$

③ Using the solution in terms of the Green's function, or the superposition principle, we have

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \, \frac{\sigma \, \delta(z'-a)}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \, \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-a)^2}}$$

$x - x' \to x'$
$y - y' \to y'$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \, \frac{1}{\sqrt{x'^2 + y'^2 + (z-a)^2}}$$

(4)

$$\phi(\vec{r}) = \frac{\sigma}{4\pi\epsilon_0} \int_0^\infty \rho' d\rho' \int_0^{2\pi} d\phi' \frac{1}{\sqrt{\rho'^2 + (z-a)^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0} 2\pi \int_0^\infty \rho' d\rho' \frac{1}{\sqrt{\rho'^2 + (z-a)^2}}$$

$$= \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} \int_0^R \frac{\rho' d\rho'}{\sqrt{\rho'^2 + (z-a)^2}}$$

→ interpret as integral over a disc of radius R.

→ otherwise, it is a divergent integral.

$$\rho'^2 + (z-a)^2 = t$$

$$2\rho' d\rho' = dt$$

$$\int \frac{dt}{2\sqrt{t}} = \frac{t^{-\frac{1}{2}+1}}{2\frac{1}{2}} = \sqrt{t}$$

$$= \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} \int_{(z-a)^2}^{R^2+(z-a)^2} \frac{dt}{2\sqrt{t}}$$

$$= \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} \left[\sqrt{R^2 + (z-a)^2} - |z-a| \right]$$

(5) For $|z-a| << R$, or $R \to \infty$

$$\phi(\vec{r}) = \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} \left[R\sqrt{1 + \frac{(z-a)^2}{R^2}} - |z-a| \right]$$

$$\approx \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} R \left[1 + \frac{1}{2}\frac{(z-a)^2}{R^2} - \frac{|z-a|}{R} + \cdots \right]$$

$$\approx \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} R \left[1 - \frac{|z-a|}{R} + O\left(\frac{|z-a|}{R}\right)^2 \right]$$

$$\approx \frac{\sigma}{2\epsilon_0} \mathop{Lt}_{R\to\infty} \left[R - |z-a| \right]$$

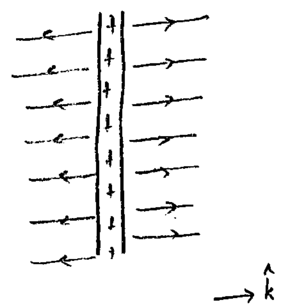
(6) Thus, the electric potential, which is the potential energy per unit charge, seems to diverge! But, one might argue that it the force that is a physical quantity. Thus, let us calculate the electric field, force per unit charge.

$$\vec{E} = -\vec{\nabla}\phi$$

$$= -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{\sigma}{2\epsilon_0}\,\underset{R\to\infty}{\mathrm{Lt}}\left[R - |z-a|\right]$$

$$= \hat{k}\,\frac{\sigma}{2\epsilon_0}\,\frac{\partial}{\partial z}|z-a|$$

$$= \begin{cases} +\dfrac{\sigma}{2\epsilon_0}\,\hat{k}, & \text{if } z>a, \\ -\dfrac{\sigma}{2\epsilon_0}\,\hat{k}, & \text{if } z<a. \end{cases}$$



(7) Thus, the electric field does come out to be finite. This is not a completely satisfactory argument, because energy of any form is a source of gravitational field, that is, if one writes the equation for gravity, energy/mass takes the place of charge in Poisson equation.

(8) We have covered spherically and planar symmetric charge distributions. The cylindrically symmetric will be discussed in the homework. In the homework we consider a finite, infinitely thin, rod of uniform charge density λ. We consider a finite rod to avoid the divergence similar to that encountered in the case of plane. We will be able to take the limit of infinite rod for electric fields. The charge density is $\lambda = \frac{\text{charge}}{\text{length}}$

$$\rho(\vec{r}) = \lambda\ \delta(x)\ \delta(y)\ \theta(-L < z < L)$$

(9)
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \frac{\delta(x')\ \delta(y')\ \theta(-L<z<L)}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$\downarrow$$

$$= \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} dz' \frac{1}{\sqrt{x^2+y^2+(z-z')^2}}$$

$\rho = \sqrt{x^2+y^2}$

$$\downarrow$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left[\sinh^{-1}\left(\frac{L-z}{\rho}\right) + \sinh^{-1}\left(\frac{L+z}{\rho}\right) \right]$$

(10) Use the identity

$$\sinh^{-1} t = \ln\left(t + \sqrt{t^2+1}\right)$$

to rewrite the expression, suitable for taking the limit $L \to \infty$, in the form

$$\phi(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0}\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L}, \frac{\rho}{L}\right)\right]$$

where

$$F(a,b) = \ln\left[1 - a + \sqrt{(1-a)^2 + b^2}\right] + \ln\left[1 + a + \sqrt{(1+a)^2 + b^2}\right]$$

(11) For $\rho << L$ and $z << L$ show that

$$\phi(\vec{r}) \approx -\frac{2\lambda}{4\pi\epsilon_0}\ln\frac{\rho}{2L}.$$

Notice the divergence for $L \to \infty$.

(12) $$\vec{E} = -\vec{\nabla}\phi = \frac{2\lambda}{4\pi\epsilon_0}\frac{\hat{\rho}}{\rho}$$

Notice how the derivative got rid of the divergence.