Date : 2014 Oct 17

1 Analogy with gravity :

$$\vec{E}(\vec{r}) \longleftrightarrow \vec{g}(\vec{r}) \qquad \rho_e(\vec{r}) \longleftrightarrow \rho_m(\vec{r}) \qquad \frac{1}{4\pi\epsilon_0} \leftrightarrow G$$

$$F_e = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \qquad F_g = \frac{G\, m_1 m_2}{r^2}$$

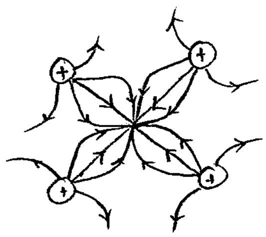
2 For weak gravitational fields, Einstein's equation reduce to the Poisson equation

$$-\nabla^2 \phi_e = \frac{1}{\epsilon_0}\rho_e$$

$$-\nabla^2 \phi_g = 4\pi G \rho_g$$

3 But, what are the sinks for gravitational field lines? How would you complete the field lines for four masses, if they are not supposed to intersect?

4 Electric field lines for four positive charges is



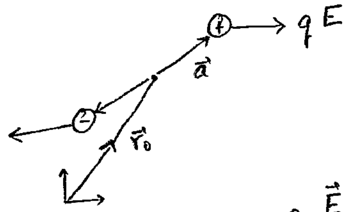
→ The field is exactly zero at the center. Thus, they do not technically intersect.

→ If, it was not zero, it would mean the lines intersect at the center, implying a negative charge. The $\vec{\nabla}\cdot\vec{E}$ at the center indeed evaluates to zero.

(5) We have until now exclusively dealt with charges (positive or negative). We would next deal with materials, which are typically neutral. Materials are made of atoms. In the presence of fluctuations the neutrality can be disturbed. In the leading order approximation it is sufficient to model the atoms to be polarizable, i.e., the positive and negative charge densities might displace their centers.

(6) To this end, let us review, what happens to an electric dipole moment $\vec{d}$ in the presence of a electric field.

(7)



$$\vec{F} = q\,\vec{E}(\vec{r}_0+\vec{a}) - q\,\vec{E}(\vec{r}_0-\vec{a})$$

$$= q\,\vec{E}(\vec{r}_0) + q\,\vec{a}\cdot\vec{\nabla}\,\vec{E}(\vec{r}_0) - q\,\vec{E}(\vec{r}_0) + q\,\vec{a}\cdot\vec{\nabla}\,\vec{E}(\vec{r}_0) + O^2$$

$$= 2q\,\vec{a}\cdot\vec{\nabla}\,\vec{E}(\vec{r}_0) \qquad \vec{d} = 2q\vec{a}$$

$$= \vec{d}\cdot\vec{\nabla}\,\vec{E}(\vec{r}_0)$$

(8) Thus, the force on an atom can be expressed as

$$\vec{F} = \vec{d}\cdot\vec{\nabla}\,\vec{E}$$
$$= -\,\vec{d}\cdot\vec{\nabla}\,\vec{\nabla}\phi$$
$$= -\,\vec{\nabla}\left(\vec{d}\cdot\vec{\nabla}\phi\right)$$
$$= \vec{\nabla}\left(\vec{d}\cdot\vec{E}\right)$$
$$= -\,\vec{\nabla}\left(-\vec{d}\cdot\vec{E}\right)$$

(using $\vec{E} = -\vec{\nabla}\phi$ → used electrostatic approximation.)

(used $\vec{d}$ to be independent of $\vec{x}$, because it is a property of an atom as a whole.)

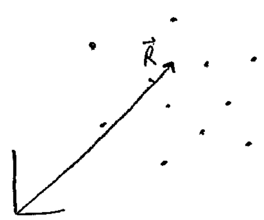
(9) Comparing the above with

$$\vec{F} = -\,\vec{\nabla}\,U_E$$

we identify the interaction energy

$$U_E = -\,\vec{d}\cdot\vec{E}$$

(10) An atom consists of many charges.



$\vec{R}$ – arbitrary point, say center of charge.

$$Q_{tot} = \sum_a q_a = 0 \quad \text{(because atom is neutral)}$$

$$\vec{d} = \sum_a q_a\left(\vec{r}_a - \vec{R}\right)$$
$$= \sum_a q_a\vec{r}_a - \vec{R}\sum_a q_a$$
$$= \sum_a q_a\vec{r}_a$$

(11) Thus, the force on an atom is

$$\vec{F}_{atom} = \sum_a q_a \vec{E}(\vec{r}_a)$$

$$= \sum_a q_a \vec{E}(\vec{R} + \vec{r}_a - \vec{R})$$

$$= \underbrace{\sum_a q_a}_{=0} \vec{E}(\vec{R}) + \underbrace{\sum_a q_a (\vec{r}_a - \vec{R})}_{\vec{d}} \cdot \vec{\nabla} \vec{E}$$

$$= \vec{d} \cdot \vec{\nabla} \vec{E}$$

(12) Force on a macroscopic material will be

$$\vec{F} = \int d^3r \; n(\vec{r}) \; \vec{F}_{atom}$$

$n(\vec{r}) = \frac{\text{atoms}}{\text{volume}}$

$= \text{number density}$

$$= \int d^3r \; \boxed{n(\vec{r}) \; \vec{d}} \cdot \vec{\nabla} \vec{E}$$

$\vec{P}(\vec{r},t) = n(\vec{r}) \, \vec{d}$

$$= \int d^3r \; \vec{P} \cdot \vec{\nabla} \vec{E}$$

$$= - \int d^3r \; (\vec{\nabla} \cdot \vec{P}) \; \vec{E}$$

(surface term does not contribute because $\vec{n}(\vec{r}) = 0$ at the surface.)

(13) Comparing this with

$$\vec{F} = q \vec{E} = \int d^3r \; \rho(\vec{r}) \; \vec{E}(\vec{r})$$

we identify the effective charge of a material as.

$$\rho_{eff}(\vec{r}) = - \vec{\nabla} \cdot \vec{P}$$