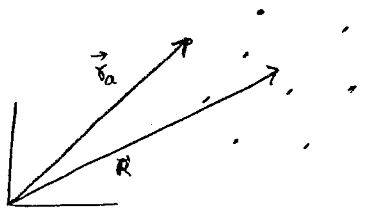
Date: 2014 Oct 20

① An atom consists of charges (electrons and protons)



$\vec{R}$ – arbitrary point, say center of charge.

$$q_{atom} = \sum_a q_a = 0$$

$$\vec{d}_{atom} = \sum_a q_a (\vec{r}_a - \vec{R})$$

$$= \sum_a q_a \vec{r}_a - \vec{R} \underbrace{\sum_a q_a}_{=0}.$$

$$= \sum_a q_a \vec{r}_a.$$

② Thus, an atom in the presence of an electric field interacts with the electric field due to its dipole moment. The interaction energy, and the corresponding force and torque, for a dipole interacting with a electric field is

$$U = - \vec{d} \cdot \vec{E}$$

$$\vec{F} = - \vec{\nabla} U = \vec{\nabla} (\vec{d} \cdot \vec{E}) = (\vec{d} \cdot \vec{\nabla}) \vec{E}$$

$$\vec{\tau} = \vec{d} \times \vec{E}$$

3 Using 1 and 2 together we have the force and torque on an individual atom to be

$$\vec{F}_{atom} = \vec{d}_{atom} \cdot \vec{\nabla}\,\vec{E}(\vec{R})$$

$$\vec{\tau}_{atom} = \vec{d}_{atom} \times \vec{E}(\vec{R})$$

4 In particular, from scratch again,

$$\vec{F}_{atom} = \sum_a q_a\,\vec{E}(\vec{r}_a)$$

$$= \underbrace{\sum_a q_a}_{=0}\,\vec{E}(\vec{R}) + \underbrace{\sum_a q_a\,(\vec{r}_a - \vec{R})}_{\vec{d}_{atom}} \cdot \vec{\nabla}\,\vec{E}(\vec{R}) + \cdots$$

$$= \vec{d}_{atom} \cdot \vec{\nabla}\,\vec{E}(\vec{R})$$

5 Force on a macroscopic material is obtained by adding (or integrating) over all the individual atoms.

$$n(\vec{r}) = \frac{\text{No. of atoms}}{\text{volume}} = \text{number density}$$

$$\vec{F} = \int d^3r\; n(\vec{r})\,\vec{F}_{atom}$$

$$= \int d^3r\; n(\vec{r})\,\vec{d}_{atom} \cdot \vec{\nabla}\,\vec{E}$$

$$= \int d^3r\; \vec{P}(\vec{r},t) \cdot \vec{\nabla}\,\vec{E}$$

$\vec{d}(\vec{r},t) \to$ distribution of atomic dipole moments, which could have time dependence.

where

$$\vec{P}(\vec{r},t) = n(\vec{r})\,\vec{d}(\vec{r},t)$$

$\vec{P}(\vec{r},t) \to$ polarization of the material.

(6) Thus, the force on a macroscopic material is given by

$$\vec{F} = \int d^3r \; \vec{P}(\vec{r},t) \cdot \vec{\nabla} \vec{E}$$

$$= \int d^3r \left(-\vec{\nabla}\cdot\vec{P}\right) \vec{E} + \underbrace{\vec{P}\,\vec{E}\Big|_{\text{surface}}}_{=0}$$

because $n(\vec{r}) = 0$ on the surface.

$$= \int d^3r \left(-\vec{\nabla}\cdot\vec{P}\right) \vec{E}$$

(7) Comparing this with

$$\vec{F} = q\vec{E} = \int d^3r \; \rho(\vec{r},t) \, \vec{E}(\vec{r},t)$$

we identify the effective charge density of a macroscopic material to be

$$\rho_{eff}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t)$$

(8) Similarly, we can show that the effective current density of a macroscopic material is given by

$$\vec{j}_{eff}(\vec{r},t) = \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}$$

Magnetization of the material

$$\vec{M}(\vec{r},t) = n(\vec{r}) \, \vec{\mu}(\vec{r},t)$$

$$\vec{\mu} = \frac{1}{2} q_a \, \vec{r}_a \times \vec{v}_a$$

(9) The Maxwell's equations in the presence of macroscopic materials reads.

$$\vec{\nabla}\cdot \epsilon_0 \vec{E} = \rho - \vec{\nabla}\cdot\vec{P}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu_0} = \frac{\partial}{\partial t}\epsilon_0\vec{E} + \vec{j} + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

which can be rewritten as.

$$\vec{\nabla}\cdot\left(\epsilon_0\vec{E} + \vec{P}\right) = \rho$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right) + \vec{j}.$$

(10) We can define

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

to write the macroscopic Maxwell's equation

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{j}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$