

- ① We have developed the machinery to consider both charges and neutral materials. Let us consider a neutral material with permanent polarization (uniformly distributed, per unit area):

$$\vec{P} = \epsilon_0 \hat{z} \theta(-z)$$



where $\epsilon_0 = \frac{\text{dipole moment}}{\text{volume}}$.

- ② Maxwell's equations tell us.

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E}) = \rho - \underbrace{\vec{\nabla} \cdot \vec{P}}_{\text{material}}$$

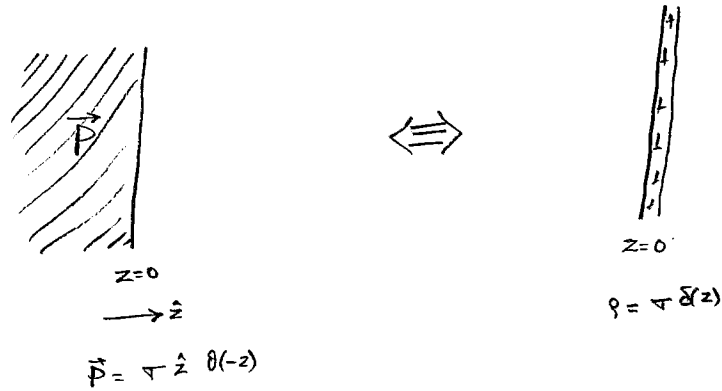
$\downarrow$ charges

- ③ The effective charge density due to material is

$$\begin{aligned} \rho_{\text{eff}}(\vec{r}) &= -\vec{\nabla} \cdot \vec{P} \\ &= -\vec{\nabla} \cdot \left[\epsilon_0 \hat{z} \theta(-z) \right] \\ &= -\epsilon_0 \hat{z} \cdot \vec{\nabla} \theta(-z) \\ &= -\epsilon_0 \frac{\partial}{\partial z} \theta(-z) \\ &= \epsilon_0 \delta(z) \end{aligned}$$

$$\hat{z} \cdot \vec{\nabla} = \frac{\partial}{\partial z}$$

- ④ Thus, effectively, the complete half slab is equivalent to an infinitely thin charged plate with surface charge density σ .



- ⑤ Thus, our equation becomes.

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

$$= \sigma \delta(z)$$

$$-\nabla^2 \phi(\vec{r}) = \frac{\sigma}{\epsilon_0} \delta(z)$$

- ⑥ Using our earlier discussion we know

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz' \frac{\sigma \delta(z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{\sigma}{2\epsilon_0} \lim_{R \rightarrow \infty} \int_0^R r dr \frac{1}{\sqrt{r^2 + z^2}}$$

⑦ Substituting $x^2 + z^2 = t$ we can evaluate

$$\phi(\vec{r}) = \frac{\sigma}{2\epsilon_0} \lim_{R \rightarrow \infty} [R - |z|],$$

which of course is divergent.

⑧ The electric field is given by

$$\begin{aligned} \vec{E} &= -\vec{\nabla} \phi(\vec{r}) \\ &= \frac{\sigma}{2\epsilon_0} \vec{\nabla} |z| \\ &= \frac{\sigma}{2\epsilon_0} \hat{z} \frac{d}{dz} |z| \end{aligned}$$

$$= \begin{cases} -\frac{\sigma}{2\epsilon_0} \hat{z}, & \text{if } z < 0 \\ +\frac{\sigma}{2\epsilon_0} \hat{z}, & \text{if } z > 0 \end{cases}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

⑨ Thus, we have.

$$\vec{P} = \begin{cases} \frac{\sigma}{2} \hat{z}, & z < 0, \\ 0, & z > 0, \end{cases} \quad \epsilon_0 \vec{E} = \begin{cases} -\frac{\sigma}{2} \hat{z}, & z < 0, \\ +\frac{\sigma}{2} \hat{z}, & z > 0, \end{cases}$$

$$\vec{D} = \begin{cases} \frac{\sigma}{2} \hat{z}, & z < 0, \\ \frac{\sigma}{2} \hat{z}, & z > 0, \end{cases}$$

