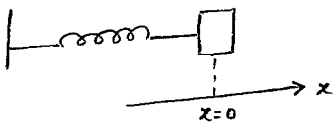
Date : 2014 Oct 31

1. Consider an harmonic oscillator, which is described by the differential equation



$$m \frac{d^2}{dt^2} x_0(t) = - m \omega^2 x_0(t) .$$

It is a homogeneous linear differential equation,

$$- \left(\frac{d^2}{dt^2} + \omega^2 \right) x_0(t) = 0 .$$

2. The differential equation also needs information about boundary / initial conditions. For a harmonic oscillator then is specified by the initial position and initial velocity, $\dot{x} \equiv \frac{dx}{dt}$.

$x(0) \longrightarrow$ initial position

$\dot{x}(0) \longrightarrow$ initial velocity.

3. Consider two independent initial conditions, which will describe two independent system,

initial condition A : $x_A(0) = 1$, $\dot{x}_A(0) = 0$

$\longrightarrow$ This is only an example and not necessary for the following proof.

initial condition B : $x_B(0) = 0$, $\dot{x}_B(0) = 1$

(4) The two independent solutions satisfy

$$-\ddot{x}_A - \omega^2 x_A = 0$$

$$-\ddot{x}_B - \omega^2 x_B = 0$$

(5) Multiplying by $x_B(t)$ and $x_A(t)$, respectively, and subtracting,

$$-x_B\ddot{x}_A - \omega^2 x_B x_A = 0$$

$$\ominus \quad -x_A\ddot{x}_B - \omega^2 x_A x_B = 0$$

$$-x_B\ddot{x}_A + x_A\ddot{x}_B - \omega^2 x_B x_A + \omega^2 x_A x_B = 0$$

$$-\frac{d}{dt}\left[x_B\frac{dx_A}{dt}\right] + \left(\frac{dx_B}{dt}\right)\left(\frac{dx_A}{dt}\right) + \frac{d}{dt}\left[x_A\frac{dx_B}{dt}\right] - \left(\frac{dx_A}{dt}\right)\left(\frac{dx_B}{dt}\right) = 0$$

$$\frac{d}{dt}\left[x_A(t)\,\dot{x}_B(t) - \dot{x}_A(t)\,x_B(t)\right] = 0$$

(6) Define the Wronskian

$$W[x_A(t), x_B(t)] = \begin{vmatrix} x_A(t) & x_B(t) \\ \dot{x}_A(t) & \dot{x}_B(t) \end{vmatrix}$$

$$= x_A(t)\,\dot{x}_B(t) - \dot{x}_A(t)\,x_B(t)$$

⑦ In terms of the Wronskian we have ⑤ as

$$\frac{d}{dt} W[x_A(t), x_B(t)] = 0$$

$$\Rightarrow \quad W[x_A(t), x_B(t)] = \text{constant in time}$$

$$= W[x_A(0), x_B(0)]$$

⑧ Note that if $x_A(t) = x_B(t)$, then $W = 0$. In general, we can conclude that

$$W[x_A(t), x_B(t)] \begin{cases} = 0, & \text{if } x_A(t) \text{ and } x_B(t) \text{ are dependent.} \\ \neq 0, & \text{if } x_A(t) \text{ and } x_B(t) \text{ are independent.} \end{cases}$$

⑨ For the homogeneous differential equation describing the harmonic oscillator,

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) x_0(t) = 0,$$

we can show that,

$$W[\sin\omega t, \cos\omega t] = -2\omega,$$

$$W[e^{i\omega t}, e^{-i\omega t}] = -2i\omega,$$

implying they are independent sets of solutions.

(10) Next, let us consider a forced harmonic oscillator, described by the differential equation (say $m=1$)

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t) = F(t).$$

Further, let us abbreviate

$$L\,x(t) = F(t).$$

The corresponding homogeneous differential equation is

$$L\,x_0(t) = 0.$$

(11) Let us say we determined a particular solution, $x_1(t)$ to the non-homogeneous differential equation, that is,

$$L\,x_1(t) = 0.$$

Notice that, then, the general solution is

$$x(t) = x_0(t) + x_1(t).$$

(12) If one worries that another particular solution, would also lead to an independent constructs, one notices

$$L\left[x_1(t) - x_2(t)\right] = 0,$$

implying the arbitrariness in the general solution, $x(t)$, is of the form $x_0(t)$.