Date : 2014 Nov 4

1 Consider the homogeneous differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) \phi_0(z) = 0 ,$$

which has solution of the form.

$$\phi_0(z) = A e^{kz} + B e^{-kz} ,$$

where A and B are determined by boundary conditions.

2 Next, consider a non-homogeneous differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) \phi(z) = \rho(z) ,$$

with boundary conditions

$$\phi(-\infty) = 0 ,$$
$$\phi(+\infty) = 0 .$$

3 To solve this non-homogeneous differential equation we construct the Green function

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z, z') = \delta(z-z')$$

with boundary conditions

$$g(-\infty, z') = 0 ,$$
$$g(+\infty, z') = 0 .$$

(4) Thus, we have.

$$\left(-\frac{d^2}{dz^2} + k^2\right)\phi(z) = \rho(z) \quad \text{— (i)}$$

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z,z') = \delta(z-z') \quad \text{— (ii)}$$

(5) Multiply by $\phi(z)$ in (4) + ii) and integrate w.r.t. z.

$$\int_{-\infty}^{+\infty} dz\ \phi(z)\left(-\frac{d^2}{dz^2} + k^2\right) g(z,z') = \int_{-\infty}^{+\infty} dz\ \phi(z)\ \delta(z-z')$$

$$\phi(z') = -\int_{-\infty}^{+\infty} dz\ \phi(z)\ \frac{d^2}{dz^2} g(z,z') + k^2\int_{-\infty}^{+\infty} dz\ \phi(z)\ g(z,z')$$

$$= +\int_{-\infty}^{+\infty} dz \left[\frac{d}{dz}\phi(z)\right]\left[\frac{d}{dz} g(z,z')\right] + k^2\int_{-\infty}^{+\infty} dz\ \phi(z)\ g(z,z')$$

$$- \underbrace{\left[\phi(z)\ \frac{d}{dz} g(z,z')\right]\Bigg|_{z=-\infty}^{z=+\infty}}_{=0}$$

$$= -\int_{-\infty}^{+\infty} dz \left[\frac{d^2}{dz^2}\phi(z)\right] g(z,z') + k^2\int_{-\infty}^{+\infty} dz\ \phi(z)\ g(z,z')$$

$$+ \underbrace{\left[\frac{d}{dz}\phi(z)\right] g(z,z')\Bigg|_{z=-\infty}^{z=+\infty}}_{=0}$$

$$= \int_{-\infty}^{+\infty} dz \left[\left(-\frac{d^2}{dz^2} + k^2\right)\phi(z)\right] g(z,z')$$

$$= \int_{-\infty}^{+\infty} dz\ \rho(z)\ g(z,z')$$

6 Thus, we have.

$$\phi(z) = \int_{-\infty}^{+\infty} dz' \, \rho(z') \, g(z', z) . \qquad \left(\text{swapped } z \leftrightarrow z'\right)$$

7 We will later show that the Green's function satisfies the reciprocity theorem

$$g(z, z') = g(z', z) .$$

8 Using 7 in 6 we can write

$$\phi(z) = \int_{-\infty}^{+\infty} dz' \, g(z, z') \, \rho(z') ,$$

which is a more popular way to present the solution in terms of the Green's function.

9 The solution to 3, as we shall explicitly derive in the next class, is

$$g(z, z') = \frac{1}{2k} e^{-k|z-z'|} .$$

Let us verify that this is indeed a solution by substituting it in 3.

(10)

$$\frac{d}{dz} g(z,z') = \frac{1}{2k} \frac{d}{dz} e^{-k|z-z'|}$$

$$= -\frac{k}{2k} e^{-k|z-z'|} \frac{d}{dz}|z-z'|$$

$$= -\frac{k}{2k} e^{-k|z-z'|} \left[ \theta(z-z') - \theta(z'-z) \right]$$

$$\frac{d^2}{dz^2} g(z,z') = \frac{k^2}{2k} e^{-k|z-z'|} \underbrace{\left[ \theta(z-z') - \theta(z'-z) \right]^2}_{=1}$$

$$- \frac{k}{2k} e^{-k|z-z'|} \underbrace{\frac{d}{dz}\left[ \theta(z-z') - \theta(z'-z) \right]}_{2\delta(z-z')}$$

$$= k^2 g(z,z') - e^{-k|z-z'|} \delta(z-z') \qquad \left( \delta(z) f(z) = \delta(z) f(0) \right)$$

$$= k^2 g(z,z') - \delta(z-z')$$

$$\Rightarrow \left( -\frac{d^2}{dz^2} + k^2 \right) g(z,z') = \delta(z-z')$$

Thus, the solution in (9) does indeed satisfy the differential equation in (4).