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① We considered the non-homogeneous differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right)\phi(z) = \rho(z)$$

with boundary conditions

$$\phi(-\infty) = 0,$$
$$\phi(+\infty) = 0.$$

② To this end, we introduced the Green function equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z,z') = \delta(z-z')$$

with boundary conditions

$$g(-\infty, z') = 0,$$
$$g(+\infty, z') = 0.$$

③ We will show the following.

(i) Reciprocity theorem : $g(z,z') = g(z',z)$.

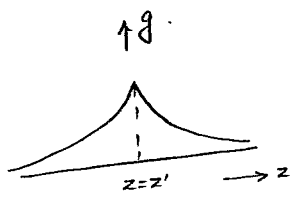
(ii) $\phi(z) = \int_{-\infty}^{+\infty} dz'\, g(z,z')\, \rho(z')$

(iii) Continuity conditions

$\rightarrow$ $$g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0$$

$\rightarrow$ $$\left[\frac{\partial}{\partial z} g(z,z')\right]\Big|_{z=z'-\delta}^{z=z'+\delta} = -1$$

(iv) Solution : $$g(z,z') = \frac{1}{2k} e^{-k|z-z'|}$$



(4) To derive the continuity condition we consider.

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z') = \delta(z-z') \quad \text{— (i)} \quad \times\, g(z,z'')$$

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z'') = \delta(z-z'') \quad \text{— (ii)} \quad \times\, g(z,z')$$

(5) Multiplying $g(z,z'')$ in (4)-(i) and $g(z,z')$ in (4)-(ii) and subtracting, after integrating w.r.t. z, we have.

$$g(z',z'') - g(z'',z') = -\int_{-\infty}^{+\infty} dz\; g(z,z'')\frac{\partial^2}{\partial z^2} g(z,z') + \int_{-\infty}^{+\infty} dz\; g(z,z')\frac{\partial^2}{\partial z^2} g(z,z'')$$

$$= -\int_{-\infty}^{+\infty} dz\, \frac{\partial}{\partial z}\Big[\underbrace{g(z,z'')\frac{\partial}{\partial z} g(z,z')}_{=0} - \underbrace{g(z,z')\frac{\partial}{\partial z} g(z,z'')}_{=0}\Big] - \int_{-\infty}^{+\infty} dz\left[\frac{\partial}{\partial z} g(z,z')\right]\left[\frac{\partial}{\partial z} g(z,z'')\right] \checkmark$$

$$+ \int_{-\infty}^{+\infty} dz\left[\frac{\partial}{\partial z} g(z,z'')\right]\left[\frac{\partial}{\partial z} g(z,z')\right] \checkmark$$

$$= 0$$

$$\Rightarrow \quad g(z',z'') = g(z'',z')$$

Note that, unlike what we have used here, the reciprocity theorem can be proved without invoking the boundary condition.

6 To derive the first continuity condition, we integrate the Green function equation with respect to z, about $z = z'$.

$$\int_{z=z'-\delta}^{z=z'+\delta} dz \left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z') = \int_{z'-\delta}^{z'+\delta} dz\, \delta(z-z')$$

$$-\left[\frac{\partial}{\partial z} g(z,z')\right]\Bigg|_{z=z'-\delta}^{z=z'+\delta} + k^2 \int_{z'-\delta}^{z'+\delta} dz\, g(z,z') = 1$$

For now, argue this term goes to zero. (second continuity equation.)

$$\left[\frac{\partial}{\partial z} g(z,z')\right]\Bigg|_{z=z'-\delta}^{z=z'+\delta} = -1.$$

7 To derive the second condition, we integrate the Green function equation after multiplying by z.

$$\int_{z'-\delta}^{z'+\delta} dz\; z \left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z') = \int_{z'-\delta}^{z'+\delta} dz\; z\, \delta(z-z')$$

$$-\int_{z'-\delta}^{z'+\delta} dz\; z\, \frac{\partial^2}{\partial z^2} g + k^2 \int_{z'-\delta}^{z'+\delta} dz\; z\, g = z'$$

goes to zero in the limit $\delta \to 0$.

8 Integrating by parts we have.

$$- z \left(\frac{\partial}{\partial z} g\right)\Bigg|_{z=z'-\delta}^{z=z'+\delta} + \int_{z'-\delta}^{z'+\delta} dz \underbrace{\left(\frac{\partial}{\partial z} z\right)}_{=1} \left(\frac{\partial}{\partial z} g\right) = z'.$$

$$- z' \underbrace{\left[\frac{\partial}{\partial z} g\right]\Bigg|_{z=z'-\delta}^{z=z'+\delta}}_{\substack{=-1 \\ \text{using } 6}} + \int_{z'-\delta}^{z'+\delta} dz \frac{\partial}{\partial z} g = z'$$

$$z' + g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = z'$$

$$g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0,$$

which is the second continuity condition.

9 Let us next explicitly solve for the Green function using

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z') = \delta(z-z')$$

using the boundary conditions

(i) $g(-\infty, z') = 0$

(ii) $g(+\infty, z') = 0$

and the continuity conditions

(i) $g(z'+\delta, z') = g(z'-\delta, z')$

(ii) $\frac{\partial}{\partial z} g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1$

(10) For $z \neq z'$ we have.

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g(z,z') = 0,$$

which has solutions

$$g(z,z') = \begin{cases} A\, e^{kz} + B\, e^{-kz}, & \text{for } z < z', \\ C\, e^{kz} + D\, e^{-kz}, & \text{for } z' < z. \end{cases}$$

(11) $g(-\infty, z') = 0 \quad \Rightarrow \quad B = 0$

and

$g(+\infty, z') = 0 \quad \Rightarrow \quad C = 0$

Thus,

$$g(z,z') = \begin{cases} A\, e^{kz}, & \text{for } z<z', \\ D\, e^{-kz}, & \text{for } z'<z. \end{cases}$$

(12) Using continuity conditions we have

$$D\, e^{-kz'} - A\, e^{kz'} = 0,$$

$$D\, e^{-kz'} + A\, e^{kz'} = \frac{1}{k},$$

which determines

$$D = \frac{1}{2k}\, e^{kz'}$$

$$A = \frac{1}{2k}\, e^{-kz'}$$

13 Using 12 in 11 we have

$$g(z,z') = \begin{cases} \frac{1}{2k} e^{-k(z'-z)}, & \text{for } z<z', \\ \frac{1}{2k} e^{-k(z-z')}, & \text{for } z'<z, \end{cases}$$

which can be compactly expressed in the form

$$g(z,z') = \frac{1}{2k} e^{-k|z-z'|}.$$