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① Maxwell's equations, for electrostatics, in the presence of charges and dielectric materials is

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \rho - \vec{\nabla} \cdot \vec{P}$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} \phi$$

② For dielectric materials that have polarization responses linear, we can write

$$\epsilon(\vec{r}) \, \vec{E} = \epsilon_0 \vec{E} + \vec{P},$$

(assumes $\vec{P}$ is linearly dependent on $\vec{E}$)

where the position dependence also encodes the geometry of the dielectric constant $\epsilon(\vec{r})$

③ Thus, we have

$$\vec{\nabla} \cdot \epsilon(\vec{r}) \, \vec{E}(\vec{r}) = \rho(\vec{r})$$

and

$$\vec{E}(\vec{r}) = -\vec{\nabla} \phi(\vec{r})$$

(4) Using one in the other, we have the differential equation relevant for electrostatics, in the presence of dielectric materials and charges;

$$-\vec{\nabla}\cdot\Big[\underbrace{\epsilon(\vec{r})}_{\text{dielectric material}}\;\vec{\nabla}\,\phi(\vec{r})\Big] = \underbrace{\rho(\vec{r})}_{\text{charge distribution.}}$$

(5) The corresponding Green's function equation is

$$-\vec{\nabla}\cdot\Big[\epsilon(\vec{r})\;\vec{\nabla}\,G(\vec{r},\vec{r}')\Big] = \delta^{(3)}(\vec{r}-\vec{r}')$$

(6) Let us consider the case of planar geometry for dielectric materials,

$$\epsilon(\vec{r}) = \epsilon(z).$$

This means we have translational symmetry in the x-y plane. This suggests we can use Fourier transformation in the x and y variables.

⑦ Let us use Fourier transformation to write

$$G(\vec{r},\vec{r}') = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, g(z,z';k_\perp)$$

$\vec{k}_\perp = k_x\hat{x} + k_y\hat{y}$

$\vec{r}_\perp = x\hat{x} + y\hat{y}$

$$\delta^{(3)}(\vec{r}-\vec{r}') = \delta(z-z')\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}$$

⑧ Notice that

$$\vec{\nabla}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} = e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\left[\, i\vec{k}_\perp + \hat{z}\frac{\partial}{\partial z}\right]$$

and for planar geometry

$$\vec{\nabla}\cdot\epsilon(z)\vec{\nabla} = \frac{\partial}{\partial z}\,\epsilon(z)\,\frac{\partial}{\partial z} - k_\perp^2\,\epsilon(z)$$

⑨ Using ⑤ to ⑧ we thus have.

$$G(\vec{r},\vec{r}') = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, g(z,z';k_\perp)$$

where the reduced Green's function satisfies the equation

$$\left[-\frac{\partial}{\partial z}\,\epsilon(z)\,\frac{\partial}{\partial z} + k_\perp^2\,\epsilon(z)\right] g(z,z';k_\perp) = \delta(z-z')$$

(10) Let us consider the trivial case of $\epsilon(z) = \epsilon_0$. Then,

$$\left(-\frac{\partial^2}{\partial z^2} + k_\perp^2 \right) \epsilon_0 \, g(z, z'; k_\perp) = \delta(z - z'),$$

whose solution is

$$g(z, z'; k_\perp) = \frac{1}{\epsilon_0} \frac{1}{2k_\perp} e^{-k_\perp |z - z'|}$$

(11) Thus, the total free Green's function is, using (10) in (9), for $\epsilon(z) = \epsilon_0$,

$$G_0(\vec{r}, \vec{r}\,') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r} - \vec{r}\,')_\perp} \frac{1}{\epsilon_0} \frac{1}{2k_\perp} e^{-k_\perp |z - z'|},$$

which is the electric potential due to a unit point charge. Thus,

$$\frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}\,'|} = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r} - \vec{r}\,')_\perp} \frac{1}{\epsilon_0} \frac{1}{2k_\perp} e^{-k_\perp |z - z'|}$$

12 We have, uptil now, derived three representation for the free Green's functions:

$$G_0(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{\epsilon_0}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,\frac{1}{k^2}$$

→ suitable when translation symmetry exists in all space directions

$$= \frac{1}{\epsilon_0}\int\frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')}\,\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

→ suitabe when translational symmetry exists in two spacial directions.

13 As we go on we will collect other representations, suitable for specific geometries. In particular for cylindrical and spherical geometries.