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1. Maxwell's equations, for electrostatics, reduces to finding the electric potential,

$$- \vec{\nabla} \cdot \left[\epsilon(\vec{r}) \, \vec{\nabla} \phi(\vec{r}) \right] = \rho(\vec{r}) .$$

dielectric materials. charge distribution

2. The Green function corresponding to this is

$$- \vec{\nabla} \cdot \left[\epsilon(\vec{r}) \, \vec{\nabla} G(\vec{r}, \vec{r}') \right] = \delta^{(3)}(\vec{r} - \vec{r}')$$

3. For planar geometry,

$$\epsilon(\vec{r}) = \epsilon(z),$$

we have.

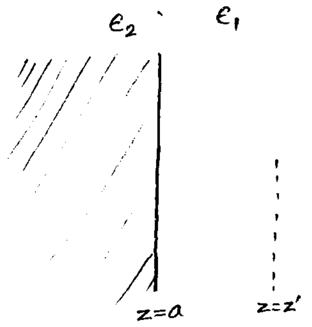
$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

$$G(\vec{r}, \vec{r}') = \int \frac{d^2k_\perp}{(2\pi)^2} \, e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \, g_\epsilon(z, z'; k_\perp)$$

where the reduced Green's function $g(z, z'; k_\perp)$ satisfies

$$\left[- \frac{\partial}{\partial z} \epsilon(z) \frac{\partial}{\partial z} + k_\perp^2 \epsilon(z) \right] g_\epsilon(z, z'; k_\perp) = \delta(z - z') .$$

(4) Let us consider the situation



$$\epsilon(z) = \begin{cases} \epsilon_2 , & z < a , \\ \epsilon_1 , & z > a . \end{cases}$$

(5) For $a < z'$, say, we can look for solutions of the following regions the kind, because in each of the $\epsilon(z)$ is uniform and the δ-function does not contribute. Thus,

$$g_\epsilon(z,z';k_\perp) = \begin{cases} A\, e^{k_\perp z} + B\, e^{-k_\perp z} , & z < a < z' , \\ C\, e^{k_\perp z} + D\, e^{-k_\perp z} , & a < z < z' , \\ E\, e^{k_\perp z} + F\, e^{-k_\perp z} , & a < z' < z . \end{cases}$$

6 The constraints that these solutions have to satisfy are.

(i) $g(-\infty, z'; k_\perp) = 0$

(ii) $g(+\infty, z'; k_\perp) = 0$

(iii) $g(z, z'; k_\perp) \Big|_{z=z'-\delta}^{z=z'+\delta} = 0$

(iv) $\epsilon(z) \frac{\partial}{\partial z} g(z, z'; k_\perp) \Big|_{z=z'-\delta}^{z=z'+\delta} = -1$ → notice the $\epsilon(z)$ factor.

(v) $g(z, z'; k_\perp) \Big|_{z=a-\delta}^{z=a+\delta} = 0$

(vi) $\epsilon(z) \frac{\partial}{\partial z} g(z, z'; k_\perp) \Big|_{z=a-\delta}^{z=a+\delta} = 0$

7 The six conditions can be used to determine the six unknowns in 5, as will be described in homework, to obtain

$$g_e(z, z'; k_\perp) = \begin{cases} \dfrac{1}{\epsilon_2} \dfrac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \dfrac{1}{\epsilon_2} \dfrac{1}{2k_\perp} \dfrac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{-k_\perp|z-a|} e^{-k_\perp|z'-a|}, & z' < a. \\ \dfrac{1}{\epsilon_1} \dfrac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \dfrac{1}{\epsilon_1} \dfrac{1}{2k_\perp} \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_\perp|z-a|} e^{-k_\perp|z'-a|}, & a < z' \end{cases}$$

8. Interpreting the second term as the contribution due to a "image charge" we identify the magnitude of the image charge as.

$$q_{image} = \begin{cases} \dfrac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}, & \text{if } z' < a, \\[2ex] \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}, & \text{if } a < z'. \end{cases}$$

and the position of the image to be

$$\left|\vec{r} - \vec{r}\,'_{image}\right| = (x - x')\,\hat{i} + (y - y')\,\hat{j} + \Big|\,|z-a| + |z'-a|\,\Big|\,\hat{k}$$

or

$$\left|z - z'_{image}\right| = \Big|\,|z-a| + |z'-a|\,\Big|,$$

which implies.

$$z'_{image} = \begin{cases} 2a - z', & \text{if } z' < a,\ z < a, \\ z', & \text{if } z' < a,\ a < z, \\ z', & \text{if } a < z',\ z < a, \\ 2a - z', & \text{if } a < z',\ a < z'. \end{cases}$$

Thus, the image charge is located behind the "mirror" if the observation point is on the same side as the charge.