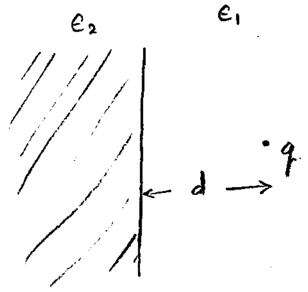
Date: 2014 Nov 14

(1) As an application of the Green's function we derived in the last class let us find the electric field due to a point charge placed in front of a dielectric material.



(2) The electric potential is given in terms of the Green function

$\vec{r}' \to$ position of charge q. Say, $0\hat{i} + 0\hat{j} + d\hat{k}$.

$$\phi(\vec{r}) = q\, G(\vec{r}, \vec{r}'),$$

which in terms of the Green function we found is

$$\phi(\vec{r}) = q \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}')_\perp}\, g(z, z'; k_\perp)$$

$$= \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp \cdot (\vec{r}-\vec{r}')_\perp} \left[\frac{q}{\epsilon_1} \frac{1}{2k_\perp} e^{-k_\perp |z-z'|} + \frac{q}{\epsilon_1} \frac{1}{2k_\perp} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} e^{-k_\perp |z|} e^{-k_\perp |z'|} \right]$$

③ Comparison with the expression for the electric potential for a point charge in vacuum,

$$\phi_0(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} = \int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, \frac{q}{\epsilon_0}\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|},$$

we conclude that the configuration of a charge in front of a dielectric plate can be thought of as a charge and another "image charge".

④ This is the context of method of images. In particular, reading off from ②, we identify

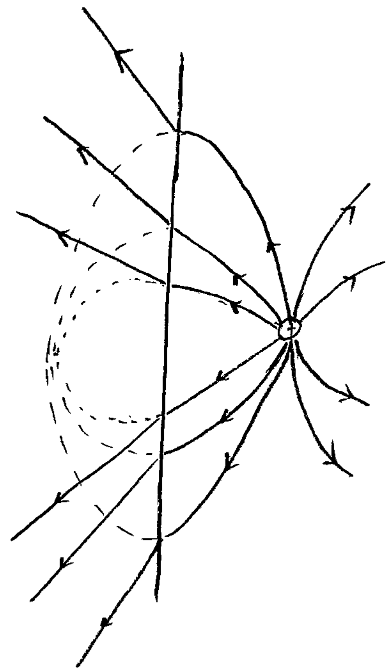
$$q_{image} = q\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}$$

$$z'_{image} = \begin{cases} d, & \text{if } 0<z \\ -d, & \text{if } z<0 \end{cases}$$

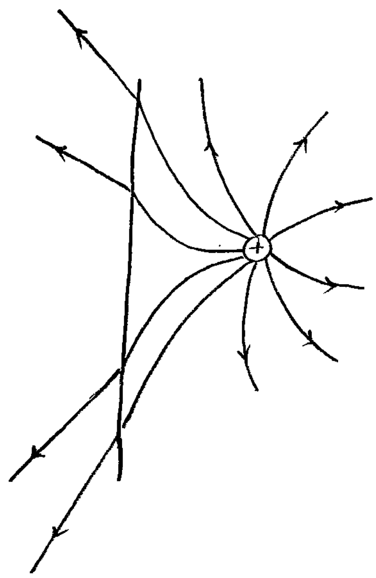
⑤ Thus,

$$\phi(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1}\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}} + \dfrac{q}{4\pi\epsilon_1}\dfrac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\dfrac{1}{\sqrt{x^2+y^2+(z+d)^2}}, & 0<z \\[2ex] \dfrac{q}{4\pi\epsilon_1}\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}} + \dfrac{q}{4\pi\epsilon_1}\dfrac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}}, & z<0 \end{cases}$$

(b) $\epsilon_2 > \epsilon_1$



$\epsilon_2 < \epsilon_1$



(6) The electric field is calculated by taking the gradient of the potential. Using the expression for the electric field of a point charge in vacuum,

$$\vec{E}_0(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

we have the electric field for a point charge in front of a flat dielectric plate given by

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1} \dfrac{x\hat{i} + y\hat{j} + (z-d)\hat{k}}{\left[x^2 + y^2 + (z-d)^2\right]^{\frac{3}{2}}} + \dfrac{q}{4\pi\epsilon_1} \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \dfrac{x\hat{i} + y\hat{j} + (z+d)\hat{k}}{\left[x^2 + y^2 + (z+d)^2\right]^{\frac{3}{2}}}, & 0 < z \\[2ex] \dfrac{q}{4\pi} \dfrac{2}{\epsilon_1 + \epsilon_2} \dfrac{x\hat{i} + y\hat{j} + (z-d)\hat{k}}{\left[x^2 + y^2 + (z-d)^2\right]^{\frac{3}{2}}}, & z < 0 \end{cases}$$

(7) To study the continuity, or discontinuity, in electric field we shall calculate the components of

$$\frac{E_x(x, y, +\delta)}{E_x(x, y, -\delta)} = ?,$$

$$\frac{E_z(x, y, +\delta)}{E_z(x, y, -\delta)} = ?,$$

and

$$\frac{E_y(x, y, +\delta)}{E_y(x, y, -\delta)} = ?.$$

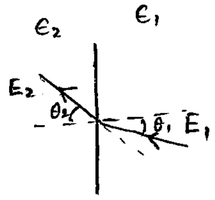
(8) Using (6) we evaluate the continuity condition to be,

$$\underset{\delta\to 0}{Lt}\ \frac{E_x(x,y,+\delta)}{E_x(x,y,-\delta)} = \frac{1+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}}{\left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right)} = 1,$$

$$\underset{\delta\to 0}{Lt}\ \frac{E_y(x,y,+\delta)}{E_y(x,y,-\delta)} = \frac{1+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}}{\left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right)} = 1,$$

$$\underset{\delta\to 0}{Lt}\ \frac{E_z(x,y,+\delta)}{E_z(x,y,-\delta)} = \frac{-d+\left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)d}{-d\left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right)} = \frac{\epsilon_2}{\epsilon_1}.$$

(9) This allows us to derive Snell's law for dielectric materials of electric field at the interface

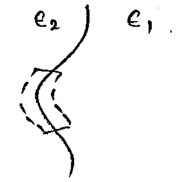


$$\tan\theta_1 = \frac{E_x(x,y,+\delta)}{E_z(x,y,+\delta)}, \qquad \tan\theta_2 = \frac{E_x(x,y,-\delta)}{E_z(x,y,-\delta)},$$

which implies.

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_2}{\epsilon_1}.$$

(10) These continuity conditions can be derived from the Maxwell equations for an arbitrary shaped interface.



$$\vec{\nabla} \cdot \left[\epsilon(\vec{r}) \, \vec{E}(\vec{r}) \right] = \rho(\vec{r})$$

Integrate on both sides over a volume of infinitely small thickness around the interface. Then using divergence theorem

$\hat{n} \rightarrow$ normal to surface.

$E'' = \vec{E} \cdot \hat{n}$

$$\epsilon_2 E_2'' = \epsilon_1 E_1''$$

Similarly, starting from

$$\vec{\nabla} \times \vec{E} = 0$$

and integrating over a surface around the interface we can derive.

$\vec{E}^{\perp} = \vec{E} - E'' \hat{n}$

$$\vec{E}_2^{\perp} = \vec{E}_1^{\perp} .$$