Date. 2014 Nov 17

① For two charges we have the Coulomb interaction energy

$$U = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

$\cdot q_1$ $\quad$ $\cdot q_2$

where $\vec{r}_1$ and $\vec{r}_2$ are positions of the charges.

② The interaction energy for three charges is

$$U_{int} = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|}\right]$$

$$= \frac{1}{2} \sum_{i=1}^{3} \sum_{\substack{j=1 \\ i \neq j}}^{3} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

where $i \neq j$: removes the self interaction terms

$\frac{1}{2}$: removes the double counting

③ For a continuous charge distribution the above generalizes to

$$U_{int} = \frac{1}{2}\int d^3r \int d^3r' \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} - E_{self}$$

$$= \frac{1}{2}\int d^3r \int d^3r' \rho(\vec{r})\, G_0(\vec{r}, \vec{r}\,')\, \rho(\vec{r}\,') - E_{self},$$

where E_{self} is the self energy, which does not contribute to force. Note that $G_0(\vec{r}, \vec{r}\,')$ is the free Green's function.

(4) For example, for two point charges

$$\rho(\vec{r}) = q_1\,\delta^{(3)}(\vec{r}-\vec{r}_1) + q_2\,\delta^{(3)}(\vec{r}-\vec{r}_2)$$

we have

$$U_{int} = \frac{1}{2}\int d^3r \int d^3r' \,\frac{1}{4\pi\epsilon_0}\left[q_1\,\delta^{(3)}(\vec{r}-\vec{r}_1) + q_2\,\delta^{(3)}(\vec{r}-\vec{r}_2)\right]\frac{1}{|\vec{r}-\vec{r}'|}\left[q_1\,\delta^{(3)}(\vec{r}'-\vec{r}_1) + q_2\,\delta^{(3)}(\vec{r}'-\vec{r}_2)\right]$$

$$- E_{self}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|}$$

(5) The interaction energy between a charge and a dielectric material is given by

$$U_{int} = \frac{1}{2}\int d^3r \int d^3r' \,\rho(\vec{r})\left[\underbrace{G(\vec{r},\vec{r}')}_{\text{Green's function including the dielectric material.}} - \underbrace{G_0(\vec{r},\vec{r}')}_{\text{free Green's function}}\right]\rho(\vec{r}') .$$

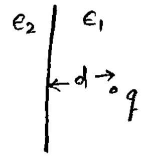
(6) From earlier lectures we have the free Green's function

$$G_0(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\,\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

and the Green's function for a dielectric slab (occupying half space)

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_1}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\left[\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|} + \frac{1}{2k_\perp}\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\, e^{-k_\perp|z|}\, e^{-k_\perp|z'|}\right]$$

7 Thus, the interaction energy of a point charge and a dielectric plate is



$$
\begin{aligned}
U_{int} &= \frac{1}{2}\int d^3r \int d^3r' \; \rho(\vec{r}) \left[G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}') \right] \rho(\vec{r}') \\
&= \frac{1}{2}\int d^3r \int d^3r' \; q\, \delta^{(3)}(\vec{r}-\vec{r}_0) \left[G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}') \right] q\, \delta^{(3)}(\vec{r}-\vec{r}_0) \\
&= \frac{q^2}{2}\left[G(\vec{r}_0,\vec{r}_0) - G_0(\vec{r}_0,\vec{r}_0) \right] \\
&= \frac{q^2}{2}\frac{1}{\epsilon_1}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp \cdot 0}\, \frac{1}{2k_\perp}\, \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\, e^{-k_\perp d}\, e^{-k_\perp d} \\
&= \frac{q^2}{16\pi^2}\frac{1}{\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\int_0^\infty \cancel{k_\perp}\, dk_\perp \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \frac{1}{\cancel{k_\perp}}\, e^{-2k_\perp d} \\
&= \frac{q^2}{8\pi}\frac{1}{\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{2d}\underbrace{\int_0^\infty dx\, e^{-x}}_{=1} . \\
&= \frac{1}{4\pi\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{q^2}{4d}
\end{aligned}
$$

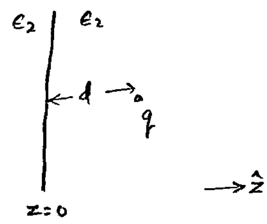
8 $$F = -\frac{\partial}{\partial d} U_{int} = \frac{1}{4\pi\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{q^2}{(2d)^2}$$

which is consistent with the image interpretation:

$$q_{image} = q\,\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}$$

distance $= 2d$.

(9) The force between the charge and the dielectric can also be calculated from the force on the induced electric charge on the dielectric plate,



$$F = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \sigma(x,y)\; E_z(x,y,0)$$

where $\sigma(x,y)$ is the charge induced in the dielectric due to the presence of q.

(10) We identify the expression for the charge density from $\epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

$$\begin{aligned}
\rho(\vec{r}) - \vec{\nabla}\cdot\vec{P} &= \rho - \vec{\nabla}\cdot\left[(\epsilon-\epsilon_0)\vec{E}\right] \\
&= \rho - \vec{\nabla}\cdot\left[\epsilon\vec{E}\left(1-\frac{\epsilon_0}{\epsilon}\right)\right] \\
&= \rho - \rho\left(1-\frac{\epsilon_0}{\epsilon}\right) - \epsilon\vec{E}\cdot\vec{\nabla}\left(1-\frac{\epsilon_0}{\epsilon}\right) \\
&= \rho(\vec{r})\frac{\epsilon_0}{\epsilon(\vec{r})} + \epsilon(\vec{r})\,\vec{E}\cdot\vec{\nabla}\frac{\epsilon_0}{\epsilon(\vec{r})}
\end{aligned}$$

$\vec{\nabla}\cdot(\epsilon\vec{E}) = \rho$

charge density with dielectric screening.

surface charge density

(11)

$$\begin{aligned}
\vec{\nabla}\frac{\epsilon_0}{\epsilon(\vec{r})} = \hat{z}\frac{\partial}{\partial z}\frac{\epsilon_0}{\epsilon(z)} &= \hat{z}\frac{\partial}{\partial z}\begin{cases}\dfrac{\epsilon_0}{\epsilon_2}, & z<0\\[2mm] \dfrac{\epsilon_0}{\epsilon_1}, & z>0\end{cases} \\
&= \hat{z}\frac{\partial}{\partial z}\left[\frac{\epsilon_0}{\epsilon_2}\theta(-z) + \frac{\epsilon_0}{\epsilon_1}\theta(z)\right] \\
&= \hat{z}\,\epsilon_0\left(\frac{1}{\epsilon_1}-\frac{1}{\epsilon_2}\right)\delta(z) \\
&= \hat{z}\,\frac{\epsilon_0}{\epsilon_1}\,\frac{\epsilon_2-\epsilon_1}{\epsilon_2}\,\delta(z)
\end{aligned}$$

$\epsilon_0 = \epsilon_1$ for our situation.

(12) Using (11) in (10) we have

$$\epsilon(\vec{r})\ \vec{E}\cdot\vec{\nabla}\frac{\epsilon_0}{\epsilon(\vec{r})} = \epsilon(z)\ E_z(x,y,z)\ \frac{\epsilon_2-\epsilon_1}{\epsilon_2}\ \delta(z)$$

$$= \underbrace{\epsilon(+\delta)}_{\epsilon_1}\ E_z(x,y,+\delta)\ \frac{\epsilon_2-\epsilon_1}{\epsilon_2}\ \delta(z)$$

Note $D_z = \epsilon E_z$ is continuous across the slab. Thus, we can use any side to evaluate it

and we identify the surface charge density,

$$\sigma(x,y) = \frac{\epsilon_1}{\epsilon_2}(\epsilon_2-\epsilon_1)\ E_z(x,y,+\delta)$$

(13) Using the surface charge density of (12) in (9) we have the force on the dielectric

$$F = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\ \left[\frac{\epsilon_1}{\epsilon_2}(\epsilon_2-\epsilon_1)\ E_z(x,y,+\delta)\right]\ E_z(x,y,0)$$

(14) The electric field, z-component, at $z=0$, is interpreted as.

$\epsilon_2 E(-\delta) = \epsilon_1 E(+\delta)$

$$E_z(x,y,0) = \frac{1}{2}\left[\ E_z(x,y,+\delta) + E_z(x,y,-\delta)\right]$$

$$= \frac{1}{2}\left[\ E_z(x,y,+\delta) + \frac{\epsilon_1}{\epsilon_2}\ E_z(x,y,+\delta)\right]$$

$$= \frac{\epsilon_1+\epsilon_2}{2\epsilon_2}\ E_z(x,y,+\delta)$$

15 Using 14 in 13 we hae.

$$F = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \frac{\epsilon_1}{\epsilon_2} (\epsilon_2 - \epsilon_1) \frac{\epsilon_1 + \epsilon_2}{2\epsilon_2} \; E_2(x,y,+\delta)^2$$

16 Using eqn. 6 in lecture dated 2014 Nov 14 we hav

$$E_2(x,y,+\delta) = \frac{q}{4\pi\epsilon_1} \frac{(-d)\left[1 - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right]}{(x^2+y^2+d^2)^{\frac{3}{2}}} = -\frac{q}{4\pi\epsilon_1} \frac{2\epsilon_2}{\epsilon_1+\epsilon_2} \frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}$$

17
$$F = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \frac{\epsilon_1}{\epsilon_2} (\epsilon_2 - \epsilon_1) \frac{\epsilon_1 + \epsilon_2}{2\epsilon_2} \left[-\frac{q}{4\pi\epsilon_1} \frac{2\epsilon_2}{\epsilon_1+\epsilon_2} \frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}} \right]^2$$

$$= \frac{1}{4\pi\epsilon_1} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} \frac{q^2}{2\pi} \int_0^{\infty} \rho \, d\rho \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \frac{d^2}{(\rho^2+d^2)^3}$$

$$= \frac{1}{4\pi\epsilon_1} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} \frac{q^2}{d^2} \int_0^{\infty} \frac{x\,dx}{(1+x^2)^3}$$

$$= \frac{1}{4\pi\epsilon_1} \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2} \frac{q^2}{(2d)^2}$$

$$\int_0^{\infty} \frac{x\,dx}{(1+x^2)^3} \qquad 1+x^2 = t, \quad 2x\,dx = dt$$

$$= \int_1^{\infty} \frac{dt}{2} \frac{1}{t^3} = \frac{1}{2} \frac{1}{-2} \frac{1}{t^2}\Big|_1^{\infty} = \frac{1}{4}$$

which verifies the expression we calculated in 8. The change in sign is because the force in 8 is on the dielectric material. They satisfy Newton's third law. and the force above is on the charge

18 <u>Homework</u>: Calculate the total induced charge to be

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \sigma(x,y) = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q .$$