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1 We have constructed different representations for the Green function, appropriate for suitable geometries, in electrostatics.

$$-\vec{\nabla}\cdot\epsilon(\vec{r})\,\vec{\nabla}\,\phi(\vec{r}) = \rho(\vec{r})$$

$$-\vec{\nabla}\cdot\,\epsilon(\vec{r})\,\vec{\nabla}\,G_\epsilon(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

2 In particular, the free Green's function, $\epsilon(\vec{r}) = \epsilon_0$, is given by the following representations

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{\epsilon_0}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,\frac{1}{k^2}$$

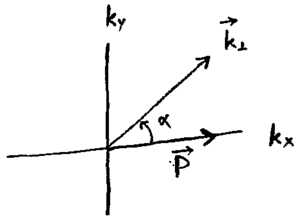
$$= \frac{1}{\epsilon_0}\int\frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\,\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

3 We shall now extend our analysis to planar geometries with rotational symmetry about an axis normal to the plane. To this end, we will introduce Bessel functions.

(4) Using the third representation in (2) let us complete the angular integration in the $\vec{k}_\perp$ - plane.

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

$$\vec{P} = (\vec{r}-\vec{r}\,')_\perp$$



→ we have chosen $\vec{P}$ to be along $\hat{i}$. Thus,

$$\vec{k}_\perp\cdot\vec{P} = k_\perp P\cos\alpha$$

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{\epsilon_0}\int_0^\infty \frac{\cancel{k_\perp}\,dk_\perp}{2\pi}\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{ik_\perp P\cos\alpha}\,\frac{1}{2\cancel{k_\perp}}\, e^{-k_\perp|z-z'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp \boxed{\int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{ik_\perp P\cos\alpha}}\; e^{-k_\perp|z-z'|}$$

(5) Bessel function of zeroth order is defined as

$$J_0(t) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}$$

⑥ To show that $J_0(t)$ is real valued, we write

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}$$

$$= \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos(t\cos\alpha) + i \int_0^{2\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha).$$

⑦
$$\int_0^{2\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) = \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) + \int_{\pi}^{2\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha)$$

↳ substitute $\alpha' = \alpha - \pi$, $d\alpha' = d\alpha$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) + \int_0^{\pi} \frac{d\alpha'}{2\pi} \sin(t\,\underbrace{\cos(\alpha'+\pi)}_{-\cos\alpha'})$$

$$= \int_0^{\pi} \frac{d\alpha}{2\pi} \sin(t\cos\alpha) - \int_0^{\pi} \frac{d\alpha'}{2\pi} \sin(t\cos\alpha')$$

$$= 0$$

⑧ Using ⑦ in ⑥ we learn that $J_0(t)$ is a real valued function,

$$\operatorname{Im} J_0(t) = 0,$$

and

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \cos(t\cos\alpha),$$

which immediately implies (because $\cos(-x) = \cos x$.)

$$J_0(-t) = J_0(t)$$

9 The free Green's function satisfies the Laplacian, in the form

$$\epsilon_0 \nabla^2 G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}') .$$

Thus, for $\vec{r} \neq \vec{r}'$, and choosing $\vec{r}' = 0$ for simplicity we have

$$\epsilon_0 \nabla^2 G(\vec{r}, 0) = 0 , \qquad \text{for } \vec{r} \neq 0 .$$

10 Thus, we conclude that the Bessel function $J_0(t)$ $|\vec{r}_\perp| = \rho$ satisfies

$$\nabla^2 \left[J_0(k_\perp \rho) \, e^{-k_\perp |z|} \right] = 0 .$$

11 Using

$$\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$

and

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

we have the differential equation satisfied by $J_0(t)$ to be

$$\left[\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} + 1 \right] J_0(t) = 0$$

or

$$\left[\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} + 1 \right] J_0(t) = 0 .$$