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① Starting from the free Green's function expression

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

and using cylindrical variables to write $d^2k_\perp = k_\perp\, dk_\perp\, d\alpha$

we have

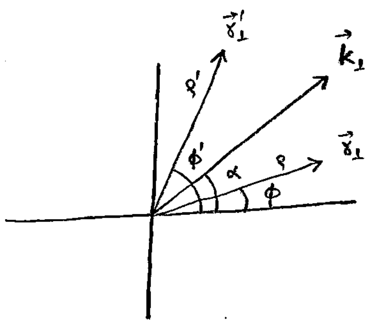
$$G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp\, e^{-k_\perp|z-z'|}\, J_0(k_\perp P)$$

where

$$J_0(k_\perp P) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}\,')_\perp}$$

$\vec{P} = (\vec{r}-\vec{r}\,')_\perp$.

$P = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$



② Bessel function of m-th order, $J_m(t)$, is defined to be Fourier components (up to i^m) of $e^{it\cos\alpha}$. Thus,

$$e^{it\cos\alpha} = \sum_{m=-\infty}^{+\infty} e^{im\alpha}\, i^m\, J_m(t),$$

$$i^m J_m(t) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{-im\alpha}\, e^{it\cos\alpha},$$

which serves as the integral representation of $J_m(t)$.

③ Addition formula for Bessel functions is the generalization of the addition formula for trigonometric functions

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'.$$

To derive the addition formula we begin with ①

$$J_0(k_\perp P) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}$$

$$= \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{i\vec{k}_\perp\cdot\vec{r}}\, e^{-i\vec{k}_\perp\cdot\vec{r}'_\perp}$$

$$= \int_0^{2\pi}\frac{d\alpha}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\alpha)} i^m J_m(k_\perp\rho)\sum_{m'=-\infty}^{+\infty} e^{-im'(\phi'-\alpha)}(-i)^{m'} J_{m'}(k_\perp\rho')$$

$$= \sum_{m=-\infty}^{+\infty}\sum_{m'=-\infty}^{+\infty} e^{im\phi-im'\phi'}\, i^m(-i)^{m'} J_m(k_\perp\rho)J_{m'}(k_\perp\rho')\underbrace{\int_0^{2\pi}\frac{d\alpha}{2\pi}\,e^{i\alpha(m'-m)}}_{\delta_{mm'}}$$

$$= \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} J_m(k_\perp\rho)\, J_m(k_\perp\rho'),$$

which is more explicitly written as.

$$J_0\left(k_\perp\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right) = \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} J_m(k_\perp\rho)\, J_m(k_\perp\rho').$$

④ The power series for Bessel function of m-th order can be derived from the integral representation.

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{-im\alpha}\, e^{it\cos\alpha} \qquad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$J_m(t) = \frac{1}{i^m} \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{-im\alpha}\, e^{i\frac{t}{2}e^{i\alpha}}\, e^{i\frac{t}{2}e^{-i\alpha}}$$

$$= \frac{1}{i^m} \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{-im\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}\left(i\frac{t}{2}e^{i\alpha}\right)^n \sum_{k=0}^{\infty} \frac{1}{k!}\left(i\frac{t}{2}e^{-i\alpha}\right)^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!}\frac{1}{k!}\, i^{n+k-m} \left(\frac{t}{2}\right)^{n+k} \underbrace{\int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{i\alpha(n-m-k)}}_{\delta_{n,m+k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(m+k)!}\frac{1}{k!}\, i^{2k} \left(\frac{t}{2}\right)^{m+2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k)!\,k!} \left(\frac{t}{2}\right)^{m+2k}$$

$$= \frac{1}{m!}\left(\frac{t}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{t}{2}\right)^{m+2} + \cdots$$