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1 We have defined Bessel's function using an integral representation,

$$J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{it\cos\alpha} \, e^{-im\alpha} .$$

It has the series representation

$$J_m(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{(n+m)!} \left(\frac{t}{2}\right)^{m+2n}$$

$$= \frac{1}{m!}\left(\frac{t}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{t}{2}\right)^{m+2} + \cdots$$

2 The free Green's function

$$G_0(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0} \sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')} \int_0^{\infty} dk_z \, J_m(k_z\rho) \, J_m(k_z\rho') \, e^{-k_z|z-z'|}$$

satisfies the Laplacian

$$-\nabla^2 G_0(\vec{r},\vec{r}') = \delta^{(3)}(\vec{r}-\vec{r}'),$$

which for $\vec{r}'=0$ and $\vec{r}\neq\vec{r}'$ is

$$\nabla^2 G_0(\vec{r},0) = 0 .$$

Thus, we have.

$$\nabla^2 \left[J_m(k_z\rho) \, e^{im\phi} \, e^{-k_z|z|} \right] = 0$$

3. Using

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

we have

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} + k_\perp^2\right] J_m(k_\perp\rho) = 0$$

which reads, using $k_\perp\rho = t$,

$$\left[\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} - \frac{m^2}{t^2} + 1\right] J_m(t) = 0$$

4. In preparation to studying the asymptotic limit of Bessel's function we note,

$$\frac{1}{\sqrt{t}}\frac{d}{dt}\sqrt{t} = \left(\frac{1}{2t} + \frac{d}{dt}\right)$$

$$\left(\frac{1}{\sqrt{t}}\frac{d}{dt}\sqrt{t}\right)^2 = \frac{1}{\sqrt{t}}\frac{d^2}{dt^2}\sqrt{t}$$

$$= \left(\frac{1}{2t} + \frac{d}{dt}\right)^2$$

$$= \frac{1}{4t^2} + \frac{1}{2t}\frac{d}{dt} + \frac{d}{dt}\frac{1}{2t} + \frac{d^2}{dt^2}$$

$$= \frac{1}{4t^2} + \frac{1}{2t}\frac{d}{dt} - \frac{1}{2t^2} + \frac{1}{2t}\frac{d}{dt} + \frac{d^2}{dt^2}$$

$$= \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{4t^2}.$$

in terms of which we can write

$$\left[\frac{1}{\sqrt{t}}\frac{d^2}{dt^2}\sqrt{t} - \frac{(m^2 - \frac{1}{4})}{t^2} + 1\right] J_m(t) = 0.$$

5 We further write

$$\left[\frac{d^2}{dt^2} - \frac{(m^2-\frac{1}{4})}{t^2} + 1\right] \sqrt{t}\, J_m(t) = 0 .$$

6 For $t \to 0$, or $|m^2 - \frac{1}{4}| >> t^2$, we have.

$$\frac{d^2}{dt^2} \sqrt{t}\, J_m(t) = \frac{(m^2-\frac{1}{4})}{t^2} \sqrt{t}\, J_m(t) .$$

Observing that, for $m>0$,

$$\frac{d^2}{dt^2} t^{m+\frac{1}{2}} = \frac{d}{dt} (m+\tfrac{1}{2})\, t^{m-\frac{1}{2}} = (m^2-\tfrac{1}{4})\, t^{m-\frac{3}{2}} = \frac{(m^2-\frac{1}{4})}{t^2} t^{m+\frac{1}{2}},$$

we find

$$\operatorname*{Lt}_{t\to 0} J_m(t) \approx t^m ,$$

which is consistant with the more accurate form, using 1,

$$\operatorname*{Lt}_{t\to 0} J_m(t) = \frac{1}{m!}\left(\frac{t}{2}\right)^m .$$

7 For $t \to \infty$, or $|m^2 - \frac{1}{4}| << t^2$, we have.

$$\left(\frac{d^2}{dt^2} + 1\right) \sqrt{t}\, J_m(t) = 0$$

which implies $\sqrt{t}\, J_m(t)$ is of the form Cos t and Sin t. This is true, and the more accurate form of this limit is

$$\underset{t\to\infty}{Lt}\ J_m(t) \approx \sqrt{\frac{\pi}{2t}}\ \cos\left(t - \left(m+\frac{1}{2}\right)\frac{\pi}{2}\right).$$

(8) Finally, we present the completeness relation for the Bessel function. We begin by noting

$$\int \frac{d^2k_\perp}{(2\pi)^2}\ e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\int_0^\infty k_\perp dk_\perp\ J_m(k_\perp\rho)\ J_m(k_\perp\rho'),$$

which is also the delta funct in two dimensions,

$$\int \frac{d^2k_\perp}{(2\pi)^2}\ e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp} = \delta^{(2)}(\vec{r}_\perp-\vec{r}_\perp') = \frac{\delta(\rho-\rho')}{\rho}\ \delta(\phi-\phi').$$

Then, using

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}$$

we obtain

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty k_\perp dk_\perp\ J_m(k_\perp\rho)\ J_m(k_\perp\rho'),$$

which is the completeness relation for Bessel functions.

(9) The orthogonality of Bessel functn is given by

$$\frac{\delta(k_\perp-k_\perp')}{k_\perp} = \int_0^\infty \rho\, d\rho\ J_m(k_\perp\rho)\ J_m(k_\perp'\rho).$$