Meeting No ③

I/ Relativistic kinematics

① 4 - position

$$x^\alpha \equiv (ct, \vec{x})$$

② 4 - velocity

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{d\vec{x}}{dt}\right)$$

$$\frac{dt}{d\tau} = \gamma$$

$$= \left(c\gamma_p, c\gamma_p\vec{\beta}_p\right)$$

③ 4 - acceleration

$$a^\alpha \equiv \frac{du^\alpha}{d\tau} = \left(c\frac{d\gamma_p}{d\tau}, c\frac{d}{d\tau}(\gamma_p\vec{\beta}_p)\right)$$

$$= \left(c\frac{dt}{d\tau}\frac{d\gamma_p}{dt}, c\frac{dt}{d\tau}\frac{d}{dt}(\gamma_p\vec{\beta}_p)\right)$$

$$= \left(c\gamma_p\frac{d\gamma_p}{dt}, c\gamma_p\frac{d}{dt}(\gamma_p\vec{\beta}_p)\right)$$

④ 4 - momentum

$$p^\alpha \equiv mu^\alpha = \left(mc\gamma_p, mc\gamma_p\vec{\beta}_p\right)$$

II Energy - momentum (4-momentum)

① $p^\alpha = m u^\alpha = (\underbrace{mc\gamma_p}_{(E/c)\ \text{Relativistic energy}}, \underbrace{mc\gamma_p \vec{\beta}_p}_{\text{Relativistic 3-momentum } (\vec{p})})$

$$= \left(\frac{mc}{\sqrt{1-\frac{v_p^2}{c^2}}}, \frac{m\vec{v}_p}{\sqrt{1-\frac{v_p^2}{c^2}}} \right)$$

$$= \left(\frac{E}{c}, \vec{p} \right)$$

② Definitions

(i) Mass by definition does not depend on velocity.

(ii) $$E \equiv \frac{mc^2}{\sqrt{1-\frac{v_p^2}{c^2}}}$$

(iii) $$\vec{p} \equiv \frac{m\vec{v}_p}{\sqrt{1-\frac{v_p^2}{c^2}}}$$

③ Mass - Energy - Momentum relation:

$$p^\alpha p_\alpha = p^0 p_0 - \vec{p}\cdot\vec{p}$$

$$= \frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = \frac{m^2c^2}{1-\frac{v^2}{c^2}} - \frac{m^2v^2}{1-\frac{v^2}{c^2}} = m^2c^2.$$

$$\boxed{E^2 = (\vec{p}\cdot\vec{p})c^2 + m^2c^4}$$

④ Rest Energy $\equiv$ mc^2

Kinetic Energy $\equiv$ $E - mc^2$

$$= \sqrt{p^2c^2 + m^2c^4} - mc^2$$

Potential Energy $\equiv$ Not yet introduced.
Depends on the kind of interaction.

Total Energy $\equiv$ $E_{tot} = R.E + K.E + P.E$

$$= E + P.E$$

⑤ Non-relativistic limit of K.E :

$$K.E = E - mc^2$$

$$= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2$$

$$= mc^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right]$$

$$= mc^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} + O\left(\frac{v^4}{c^4}\right) - 1\right]$$

$$= \frac{1}{2}mv^2 + O\left(\frac{v^4}{c^4}\right)$$

III Massless particles (Photons)

① Rigorous definitions requires quantum field theory.

② For massless spin-1 particles (Photons) we have

$$E \equiv \hbar \omega \qquad \omega = 2\pi \nu$$

$$\vec{p} \equiv \hbar \vec{k} \qquad |\vec{k}| = \frac{2\pi}{\lambda}$$

③ 4-momentum for photons

$$p^{\alpha} = \left(\frac{E}{c}, \vec{p}\right)$$

$$\Rightarrow \quad E = |\vec{p}|\, c \qquad (\text{using } m = 0)$$

$$\Rightarrow \quad \lambda \nu = c$$

④ Doppler shift ($\vec{k} \parallel \vec{k}' \parallel \vec{v} \parallel x$ axis)

$$\begin{pmatrix} \frac{\hbar\omega'}{c} \\ \hbar k' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \frac{\hbar\omega}{c} \\ \hbar k \end{pmatrix}$$

$$\frac{\hbar\omega'}{c} = \frac{\hbar\omega}{c}\gamma + \beta\gamma\hbar k$$

$$\frac{\omega'}{c} = \frac{\omega}{c}\gamma + \beta\gamma\frac{\omega}{c}.$$

$$\omega' = \omega \frac{1+\beta}{\sqrt{1-\beta^2}}$$

$$\omega' = \omega \sqrt{\frac{1+\beta}{1-\beta}}$$

IV General boost transformation

① Note that a general boost is only a subset of the Lorentz transformation.

②

$$L(\beta_x, \beta_x, \beta_z) = \begin{bmatrix} \gamma & -\gamma\beta^i \\ -\gamma\beta^i & 1 + (\gamma - 1)\dfrac{\beta^i \beta^j}{\beta^2} \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 .$$