Midterm Exam No. 01 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Date: 2014 Feb 19

1. (**10 points.**) Determine the right hand side of the following expression for all $\mathbf{r}$. (You do not need to show your work.)
$$\boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^3} = \tag{1}$$

2. (**10 points.**) The Bessel functions $J_m(t)$ are defined by the expression
$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}. \tag{2}$$
   (a) Evaluate $J_0(0)$.
   (b) Evaluate $J_m(0)$ for $m \neq 0$.

3. (**20 points.**) A point charge q is placed near a perfectly conducting plate.
   (a) Will the charge q experience a force?
   (b) If yes, calculate the force of attraction/repulsion between the charge and conducting plate when the charge is a distance a away from the plate.
   (c) If no, why not?

4. (**10 points.**) The radial part of the Green function inside a perfectly conducting cylinder of radius a is
$$g_m(\rho, \rho'; k_z) = I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{K_m(k_z a)}{I_m(k_z a)} I_m(k_z\rho) I_m(k_z\rho'), \tag{3}$$
   where $0 \leq \rho, \rho' < a$. Here $\rho_< = \text{Min}(\rho, \rho')$, $\rho_> = \text{Max}(\rho, \rho')$, k_z is the Fourier variable for the z-coordinate and m is the Fourier variable for the angular coordinate ϕ. Evaluate $g(a, \rho'; k_z)$. Give a physical reasoning for your answer.

5. (**20 points.**) The spherical harmonics are given by
$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \left(\frac{e^{i\phi}}{\sin\theta}\right)^m \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!}. \tag{4}$$
   Express $Y_{ll}(\theta, \phi)$ is terms of l, ϕ and $\sin\theta$.

6. (**10 points.**) The magnetic field inside a solenoid of radius a, and of infinite extension in the direction of the axis, is given by the expression

$$\mathbf{B}(\mathbf{r}) = \hat{\mathbf{n}}\mu_0 I n, \tag{5}$$

where n is the number of turns per unit length, I is the current, and $\hat{\mathbf{n}}$ points along the axis determined by the cross product of direction of radius vector and direction of current.

   (a) If you double the radius of the solenoid, how much does the magnetic field inside the solenoid change?

   (b) The force on a charge particle due to a magnetic field is given by $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. What is the force experienced by a charge particle q cruising on the axis of the solenoid with speed v?

7. (**20 points.**) A typical bar magnet is suitably approximated as a magnetic dipole moment $\mathbf{m}$. The vector potential for a magnetic dipole moment is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{6}$$

The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]. \tag{7}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{m} = m\,\hat{\mathbf{z}}$.

   (a) Qualitatively plot the magnetic field lines for the dipole $\mathbf{m}$. (Hint: You do not have to depend on Eq. (7) for this purpose. An intuitive knowledge of magnetic field lines should be the guide.)

   (b) Find the expression for the magnetic field on the negative z-axis. (Hint: On the negative z-axis we have, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$ and $r = z$.)