Final Exam (Spring 2014)

PHYS 420: Electricity and Magnetism II

Date: 2014 May 6

1. (**15 points.**) Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{1}$$

to derive the recurrence relation

$$2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t). \tag{2}$$

2. (**15 points.**) The magnetic field of an infinitely long straight wire carrying a steady current I is given by, (assume wire on z-axis,)

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{3}$$

where $\rho = \sqrt{x^2 + y^2}$ is the closest distance of point $\mathbf{r}$ from the wire. The Lorentz force on a particle of charge q and mass m is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{4}$$

In the absence of an electric field, qualitatively, describe the motion of a positive charge with an initial velocity in the z-direction. In particular, investigate if the particle will attain a speed in the ϕ-direction. Thus answer whether the charge will go around the wire?

3. (**20 points.**) The Maxwell equations, in vacuum, when magnetic charges and currents are present, are given by

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_e, \qquad -\boldsymbol{\nabla} \times \mathbf{E} - \mu_0 \frac{\partial}{\partial t}\mathbf{H} = \mathbf{J}_m, \tag{5a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{H} = \frac{1}{\mu_0}\rho_m, \qquad \boldsymbol{\nabla} \times \mathbf{H} - \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E} = \mathbf{J}_e. \tag{5b}$$

Without introducing potentials, derive

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \mu_0 \frac{\partial}{\partial t}\mathbf{J}_e(\mathbf{r},t) - \boldsymbol{\nabla} \times \mathbf{J}_m(\mathbf{r},t). \tag{6}$$

4. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, g(x)\, \delta(a^2x^2 - b^2). \tag{7}$$

Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left| \frac{dF}{dx}\Big|_{x=a_r} \right|}, \tag{8}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.

5. (**15 points.**) Consider the motion of a non-relativisitic particle (speed v small compared to speed of light c, $v \ll c$,) of charge q and mass m. The charge moves on a circle described by

$$\mathbf{r}(t) = \hat{\mathbf{i}}\, A \cos\omega_0 t + \hat{\mathbf{j}}\, A \sin\omega_0 t. \tag{9}$$

Find the total radiated power

$$P(t) = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{3c^3} \mathbf{a}^2(t_e), \tag{10}$$

where $\mathbf{a}(t_e)$ is the acceleration of the particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{11}$$

6. (**15 points.**) The spectral distribution of power radiated into a solid angle $d\Omega = d\phi \sin\theta d\theta$ during Čerenkov radiation, when a particle of charge q moves with uniform speed v in a medium with index of refraction

$$n = n_\varepsilon n_\mu, \qquad n_\varepsilon = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}, \qquad n_\mu = \sqrt{\frac{\mu(\omega)}{\mu_0}}, \tag{12}$$

is given by the expression

$$\frac{\partial^2 P}{\partial\omega\partial\Omega} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{n_\varepsilon^2} \frac{\omega^2 n^2}{2\pi c} \left(\frac{v^2 n^2}{c^2} - 1 \right) \delta\left(\omega - \omega \frac{vn}{c} \cos\theta \right), \tag{13}$$

where ω is the frequency of light. Čerenkov light of a given frequency is emitted on a cone of half-angle θ_c. Determine the expression for θ_c. Show that for small θ_c,

$$\theta_c \sim \sqrt{2\left(1 - \frac{c}{nv}\right)}. \tag{14}$$