Midterm Exam No. 01 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Date: 2015 Feb 11

1. (**20 points.**) The gradient operator in cylindrical coordinates (ρ, ϕ, z) is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{1}$$

The electric field of an infinitely long rod of negligible thickness is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{\rho}\hat{\boldsymbol{\rho}}, \tag{2}$$

where λ is the charge per unit length on the rod. Evaluate

$$\boldsymbol{\nabla}\cdot\mathbf{E} \quad \text{for} \quad \rho \neq 0. \tag{3}$$

Hint: The divergence of electric field at a point in space is a measure of the charge density at that point. It satisfies the Gauss's law. Further, note that

$$\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} = 0, \quad \frac{\partial}{\partial\rho}\hat{\boldsymbol{\phi}} = 0, \quad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}, \quad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}. \tag{4}$$

2. (**20 points.**) The surface charge densities on the surface of two separate and independent charged spheres are given by

$$\sigma_1(\theta,\phi) = \frac{Q}{4\pi a^2}\cos\theta, \tag{5}$$

$$\sigma_2(\theta,\phi) = \frac{Q}{4\pi a^2}\cos^2\theta, \tag{6}$$

where θ is the polar angle in spherical coordinates. Calculate the total charge on each sphere by integrating over the surface of each sphere.

3. (**20 points.**) The radial part of the Green function outside a perfectly conducting right circular cylinder of radius a is

$$g_m(\rho,\rho';k) = I_m(k\rho_<)K_m(k\rho_>) - \frac{I_m(ka)}{K_m(ka)}K_m(k\rho)K_m(k\rho'), \tag{7}$$

where $a \leq \rho, \rho' < \infty$. Here $\rho_< = \text{Min}(\rho,\rho')$, $\rho_> = \text{Max}(\rho,\rho')$, k is the Fourier variable for the z-coordinate and m is the Fourier variable for the angular coordinate ϕ. Evaluate $g_m(a,\rho';k)$. Give a physical reasoning for your answer.

4. (**20 points.**) The free Green's function represents the electric potential of a unit point charge,

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{8}$$

For a point charge placed at the origin we have, choosing $\mathbf{r}'$ at the origin, in cylindrical coordinates

$$G_0(\mathbf{r}, 0) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{\rho^2 + z^2}}. \tag{9}$$

The free Green's function also has the representation

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\varepsilon_0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} I_m(k\rho_<) K_m(k\rho_>). \tag{10}$$

Using Eq. (10), determine $G_0(\mathbf{r}, 0)$ in terms of a single integral. That is, evaluate the sum for this case.

5. (**20 points.** Take home exercise, to be submitted during exam.)
Consider a spherical cavity of radius a with perfectly conducting walls that is grounded. The inside of the cavity is described by vacuum properties ε_0. A point charge q is placed inside the cavity.

   (a) Using method of images determine the magnitude and position of the fictitious image charge that will simulate the boundary conditions of a perfect conductor on the inner surface of the conductor.

   (b) Write down the total electric potential due to the original charge (inside the sphere) and the image charge. Thus determine the electric potential everywhere inside the spherical conductor.

   (c) Determine the induced charge density on the inner surface of the spherical conductor.

   (d) Integrating the induced charge density over the inner surface of the conductor determine the total induced charge. Thus, find out if the total induced charge equals the image charge?