Midterm Exam No. 02 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Date: 2015 Mar 18

1. (**20 points.**) Given the vector differential equation

$$\boldsymbol{\nabla}\phi(\mathbf{r}) = \frac{\mathbf{r}\times(\mathbf{a}\times\mathbf{r})}{r^3} \tag{1}$$

find $\phi(\mathbf{r})$ upto a constant. Here the vector $\mathbf{a}$ is uniform (constant) with respect to $\mathbf{r}$.

2. (**20 points.**) Calculate the dipole moment

$$\mathbf{d} = \int d^3r\, \mathbf{r}\, \rho(\mathbf{r}) \tag{2}$$

of a charged spherical shell of radius a with charge density

$$\rho(\mathbf{r}) = \frac{Q}{4\pi a^2} P_1(\cos\theta)\delta(r-a). \tag{3}$$

3. (**20 points.**) A steady current I flowing through an infinitely thin wire along the x-axis is described by the current density

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{x}}\, I\delta(z)\delta(y). \tag{4}$$

When the wire is along the y-axis it is described by the current density

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{y}}\, I\delta(z)\delta(x). \tag{5}$$

The above current densities satisfy

$$\int_S d\mathbf{a}\cdot\mathbf{J} = I, \tag{6}$$

where the integration is over an open surface S that crosses the wire once. Write the current density for a wire making an angle θ with respect to the x-axis and in the x-y plane.
Hint: Verify that the current density satisfies Eq. (6) for both the x-z plane and the y-z plane.

4. (**20 points.**) Using the definition of Legendre polynomials

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{7}$$

evaluate the explicit expression (as a polynomial) for $P_5(x)$.

5. (**20 points.**) Using the definition of spherical harmonics

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}, \tag{8}$$

evaluate the explicit expressions for $Y_{21}(\theta,\phi)$ and $Y_{2,-2}(\theta,\phi)$.