Final Exam (Spring 2015)

PHYS 420: Electricity and Magnetism II

Date: 2015 May 11

1. (**20 points.**) Legendre polynomials are expressed using
$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{1}$$
Evaluate $P_3(x)$.

2. (**20 points.**) For an infinitely long wire of negligible thickness carrying a steady current I, described by
$$\mathbf{j}(\mathbf{r}) = \hat{\mathbf{z}} I \delta(x)\delta(y), \tag{2}$$
determine the magnetic field at an arbitrary point $\mathbf{r}$ using
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{3}$$

3. (**20 points.**) The electric and magnetic fields transform under a Lorentz transformation (for boost in z direction) as
$$E'_x(\mathbf{r}',t') = \gamma\, E_x(\mathbf{r},t) + \beta\gamma\, B_y(\mathbf{r},t), \tag{4a}$$
$$B'_y(\mathbf{r}',t') = \beta\gamma\, E_x(\mathbf{r},t) + \gamma\, B_y(\mathbf{r},t), \tag{4b}$$
$$E'_z(\mathbf{r}',t') = E_z(\mathbf{r},t) \tag{4c}$$
$$B'_x(\mathbf{r}',t') = \gamma\, B_x(\mathbf{r},t) - \beta\gamma\, E_y(\mathbf{r},t), \tag{5a}$$
$$E'_y(\mathbf{r}',t') = -\beta\gamma\, B_x(\mathbf{r},t) + \gamma\, E_y(\mathbf{r},t), \tag{5b}$$
$$B'_z(\mathbf{r}',t') = B_z(\mathbf{r},t), \tag{5c}$$
where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The transformed values of the coordinates and the fields are distinguished by a prime. Derive the invariance property
$$\mathbf{E}'(\mathbf{r}',t')\cdot\mathbf{B}'(\mathbf{r}',t') = \mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t). \tag{6}$$

4. (**20 points.**) Using the identity
$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{7}$$
where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate the integral (requiring the roots to be causal, that is, $t_r < t$)
$$\int_{-\infty}^{\infty} dt' \frac{\delta\left(t - t' - \frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}\right)}{\sqrt{x^2+y^2+(z-vt')^2}}. \tag{8}$$

5. (**20 points.**) Consider the motion of a non-relativisitic particle (speed v small compared to speed of light c, $v \ll c$,) of charge q and mass m. The motion of the charge is described by

$$\mathbf{r}(t) = \hat{\mathbf{i}}\, A \cos \omega_0 t + \hat{\mathbf{j}}\, A \sin \omega_0 t. \tag{9}$$

Find the total radiated power

$$P(t) = \frac{1}{4\pi\varepsilon_0} \frac{2q^2}{3c^3} \mathbf{a}^2(t_e), \tag{10}$$

where $\mathbf{a}(t_e)$ is the acceleration of the particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{11}$$

If your eye (that can sense visible light) were to observe radiation coming off many such particles with different oscillation frequency ω_0, which color would the radiation be dominated in?