Homework No. 01 (Spring 2015)

PHYS 420: Electricity and Magnetism II

Due date: Monday, 2015 Feb 2, 4.30pm

1. (**10 points.**) Using Mathematica (or your favourite graphing tool) plot $K_0(t), K_1(t), K_2(t)$ and $I_0(t), I_1(t), I_2(t)$ on the same plot. (Please do not submit hand sketched plots.)

2. (**10 points.**) Starting from
$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{1}$$
show that
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}, \tag{2}$$
where (ρ, ϕ, z) are the cylindrical coordinates.

3. (**10 points.**) Show that the integral representation for modified Bessel functions,
$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{3}$$
satisfies the differential equation for modified Bessel functions,
$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right] K_m(t) = 0. \tag{4}$$
Hint: Integrate by parts, after identifying
$$\left(t\cosh\theta - t^2\sinh^2\theta\right) e^{-t\cosh\theta} = -\frac{d^2}{d\theta^2}e^{-t\cosh\theta}. \tag{5}$$

4. (**20 points.**) Using the integral representations for the modified Bessel functions,
$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{6a}$$
$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\cos m\phi\, e^{t\cos\phi}, \tag{6b}$$
derive the asymptotic forms for large t,
$$K_m(t) \xrightarrow{1\ll t} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}, \tag{7a}$$
$$I_m(t) \xrightarrow{1\ll t} \frac{1}{\sqrt{2\pi}}\frac{e^{t}}{\sqrt{t}}. \tag{7b}$$
Hint: The contributions to the integral are dominated near the lower limit, so use $\cosh m\theta \sim 1 + \frac{1}{2}m^2\theta^2$ and $\cos m\phi \sim 1 - \frac{1}{2}m^2\phi^2$.

5. (**20 points.**) The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt}+\frac{m^2}{t^2}+1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{8}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(t)K_m'(t) - K_m(t)I_m'(t) = -\frac{C}{t}, \tag{9}$$

where

$$I_m'(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K_m'(t) \equiv \frac{d}{dt}K_m(t). \tag{10}$$

Further, determine the value of the constant C on the right hand side of Eq. (9) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{11}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{12}$$

6. (**20 points.**) Verify by substitution that

$$\begin{aligned} g_m(\rho,\rho';k) &= I_m(k\rho_<)K_m(k\rho_>) \\ &= \theta(\rho'-\rho)I_m(k\rho)K_m(k\rho') + \theta(\rho-\rho')I_m(k\rho')K_m(k\rho) \end{aligned} \tag{13}$$

satisfies the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{m^2}{\rho^2}+k^2\right]g_m(\rho,\rho';k) = \frac{\delta(\rho-\rho')}{\rho}. \tag{14}$$

Hint: Use the identity $d\theta(x)/dx = \delta(x)$.